Whittaker models
of
degenerate principal series

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$G = KAN$: a real reductive Lie group
$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$: complexifications of Lie algebras
For $\pi \in \widehat{G}_{ad}$ and $\varpi \in \hat{N}$
Whittaker model: $\pi \hookrightarrow \mathrm{Ind}_N^G \varpi$

$\Sigma(\mathfrak{g})$: the root system for the pair $(\mathfrak{g}, \mathfrak{a})$ with the Weyl group W
$\cup$
$\Sigma(\mathfrak{g})^+$: the positive root system corresponding to $\mathfrak{n}$
$\cup$
$\Psi(\mathfrak{g})$: the fundamental system
$\mathfrak{g}^\alpha := \{X \in \mathfrak{g}\,; \mathrm{ad}(H)X = \alpha(H)X \ (\forall H \in \mathfrak{a})\}$ for $\alpha \in \Sigma(\mathfrak{g})$.

For $\Theta \subset \Psi(\mathfrak{g})$ put
$W_\Theta := \langle s_\alpha\,; \alpha \in \Theta \rangle$ and $W(\Theta) := \{w \in W\,; w\Theta \subset \Sigma(\mathfrak{g})^+\}$. Then
$W(\Theta) \times W_\Theta \xrightarrow{\sim} W \ : \ (w_1, w_2) \mapsto w_1 w_2$

$P = MAN$: a minimal parabolic subgroup with $M = Z_K(\mathfrak{a})$
$P_\Theta := PW_\Theta P = M_\Theta A_\Theta N_\Theta = G_\Theta N_\Theta$.

Here $G_\Theta = M_\Theta A_\Theta$ and $\Psi(\mathfrak{g}_\Theta) = \Theta$.

Theorem. Let $\lambda \in \widehat{G}_\Theta$ and $\varpi \in \hat{N}$.
Suppose $\dim \lambda < \infty$, $\dim \varpi = 1$ and $\lambda|_{A_\Theta}$ is generic. Then

$\dim \mathrm{Hom}_{(\mathfrak{g},K)}(\mathrm{Ind}_{G_\Theta N_\Theta}^G(\lambda \otimes 1), \mathrm{Ind}_N^G \varpi) = \#W(\mathrm{supp}\varpi, \Theta) \cdot \#W_{\mathrm{supp}\varpi} \cdot \dim_M \lambda$
$\dim \mathrm{Hom}_{C^\infty}(\mathrm{Ind}_{G_\Theta N_\Theta}^G(\lambda \otimes 1), \mathrm{Ind}_N^G \varpi) = \#W(\mathrm{supp}\varpi, \Theta) \cdot \dim_M \lambda$ if ϖ is unitary

$\mathrm{supp}\varpi := \{\alpha \in \Psi(\mathfrak{g})\,;\, \varpi(\mathfrak{g}^\alpha) \neq \{0\}\}$
$W(\Upsilon, \Theta) := \{w \in W(\Upsilon) \cap W(\Theta)^{-1}\,;\, w\Sigma(\mathfrak{g}_\Upsilon) \cap \Sigma(\mathfrak{g}_\Theta) = \emptyset\}$
$\dim_M \lambda$: the dimension of the representation of M with the same highest weight of λ.

ϖ is non-degenerate $\overset{\text{def}}{\Leftrightarrow}$ $\mathrm{supp}\varpi = \Psi(\mathfrak{g})$.

"The radial parts of K-finite functions of this Whittaker model of G"
are
"those of the non-degenerate Whittaker model of $G_{\mathrm{supp}\varpi}$".

Remark. i) $W(\Theta, \Upsilon) = W(\Upsilon, \Theta)^{-1}$.
ii) $W(\Upsilon, \Theta)$ corresponds to a subset of $P_\Theta \backslash G / P_\Upsilon$.

$$\dim \mathrm{Hom}_*(\mathrm{Ind}_{P_\Theta}^G(\lambda), \mathrm{Ind}_N^G(\varpi)) = \# W(\mathrm{supp}\varpi, \Theta) \cdot \dim_M \lambda \cdot \begin{cases} \# W_{\mathrm{supp}\varpi} \\ 1 \end{cases}$$

$$W(\Upsilon, \Theta) := \{w \in W(\Upsilon) \cap W(\Theta)^{-1}\,;\, w\Sigma(\mathfrak{g}_\Upsilon) \cap \Sigma(\mathfrak{g}_\Theta) = \emptyset\}$$

with $\Upsilon = \mathrm{supp}\varpi$

Examples. 1. ϖ is trivial. ($\Rightarrow$ Imbeddings into principal series)
$\Rightarrow \mathrm{supp}\varpi = \emptyset \Rightarrow W(\mathrm{supp}\varpi) = W \Rightarrow W(\mathrm{supp}\varpi, \Theta) = W(\Theta)^{-1}$

If $\Theta = \emptyset$, then $P_\Theta = P$, $W(\Theta) = W$, $W(\mathrm{supp}\varpi, \Theta) = W$ and the imbeddings are obtained by standard intertwining operators between principal series.

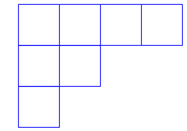
If G is compact, the result corresponds to Peter-Weyl theorem.

For general Θ they are similarly obtained through the natural imbedding of degenerate series into principal series.

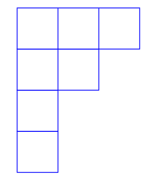
2. ϖ is non-degenerate. ($\Rightarrow$ well-studied)
$\Rightarrow W_{\mathrm{supp}\varpi} = W$, $W(\mathrm{supp}\varpi) = \{e\}$, $\Sigma(\mathfrak{g}_{\mathrm{supp}\varpi}) = \Sigma(\mathfrak{g})$.
Hence $W(\mathrm{supp}\varpi, \Theta) \neq \emptyset \Rightarrow \Theta = \emptyset$ and $W(\mathrm{supp}\varpi, \emptyset) = \{e\}$.

3. $\underline{G = GL(n, \mathbb{R})}$. Θ corresponds to a partition of n and a Young diagram:

$$G_\Theta = GL(2,\mathbb{R}) \times GL(4,\mathbb{R}) \times GL(1,\mathbb{R}) \Rightarrow 7 = 4+2+1 :$$



$\# W(\text{supp}\varpi, \Theta) = 1$
$\Leftrightarrow$ The partition suppϖ equals the dual partition of Θ



$$7 = (4+2+1)' = 3+2+1+1 \Rightarrow 2+1+3+1:$$
$$G_{\text{supp}\varpi} = GL(2,\mathbb{R}) \times GL(1,\mathbb{R}) \times GL(3,\mathbb{R}) \times GL(1,\mathbb{R})$$

$$\begin{aligned} n &= 7 \\ &= 2+4+1 \\ &= 2+1+3+1 \end{aligned} \qquad \begin{array}{lll} & & p'_3 \; p'_1 \; p'_2 \; p'_4 \\ \lambda_2 : & n'_2 = 4 & \boxed{4}\boxed{2}\boxed{3}\boxed{5} \\ \lambda_1 : & n'_1 = 2 & \boxed{1}\boxed{0} \\ \lambda_3 : & n'_3 = 1 & \boxed{6} \end{array} \qquad P_{2,4,1} = \begin{pmatrix} GL(2) & & \\ * & GL(4) & \\ * & * & GL(1) \end{pmatrix}$$

$$E_{2,4,1}(\lambda_1, \lambda_2, \lambda_3) := \text{Ind}_{P_{2,4,1}}^{G}(\lambda_1, \lambda_2, \lambda_3)$$
$$\mapsto E(\lambda_1, \lambda_2 + 1) \otimes E(\lambda_2 + 1) \otimes E(\lambda_1 - 2, \lambda_2, \lambda_3 + 1) \otimes E(\lambda_2 - 1)$$

Suppose $G = GL(n, \mathbb{R})$ and $\#W(\text{supp}\varpi, \Theta) = 1$. Then "$K$-finite functions of the Whittaker model are reduced to the usual Whittaker functions"

$\Leftrightarrow$ $G_{\text{supp}\varpi}$ is a direct product of some copies of $GL(2, \mathbb{R})$ and/or $GL(1, \mathbb{R})$

$\Leftrightarrow$ P_Θ is a maximal parabolic subgroup

$x = \left(x_{ij}\right) \in GL(n, \mathbb{R})$

$(E_{ij}\varphi)(x) = \frac{d}{dt}\varphi(xe^{tE_{ij}})|_{t=0}$, $E_{ij} = \sum_{\nu=1}^{n} x_{\nu i}\frac{\partial}{\partial x_{\nu j}}$,

$\mathfrak{n} = \sum_{1 \le j < i \le n} \mathbb{C}E_{ij}$

$\varpi\Big(\exp(\sum_{i>j} t_{ij}E_{ij})\Big) = e^{\sqrt{-1}(c_1 t_{21} + \cdots + c_{n-1}t_{n,n-1})}$

$\Theta = \{1, 2, \ldots, k-1, k+1, \ldots, n-1\}$ $(2 \le 2k \le n)$, $P_\Theta = P_{k,n-k}$

"Existence of the Whittaker model of $\text{Ind}_{P_{k,n-k}}^{G}(\lambda, \mu)$"

$\Leftrightarrow$ $c_i c_{i+1} = c_{i_1} c_{i_2} \cdots c_{i_{k+1}} = 0$ $(1 \le i < n,\ 1 \le i_1 < \cdots < i_{k+1} < n)$

For example, put

$$\begin{cases} c_i = 0 & (i = 2, 4, \ldots, 2k, 2k+1, 2k+2, \ldots, n-1), \\ c_{2j-1} \neq 0 & (j = 1, \ldots, k) \end{cases}$$

$\Rightarrow$ The K-fixed vector of the Whittaker model is the solution of

$$\begin{cases} E_i v = \mu v \qquad\qquad\qquad\qquad (i = 2k+1, 2k+2, \ldots, n), \\ \big(E_{2j-1} + E_{2j}\big) v = (\lambda + \mu - 2j + k + 1) v, \\ \Big(\big(\frac{E_{2j-1} - E_{2j}}{2}\big)^2 - \big(\frac{E_{2j-1} - E_{2j}}{2}\big) - c_{2j-1}^2 e^{2(t_{2j-1} - t_{2j})}\Big) v = \frac{\lambda - \mu - k + 1}{2}\big(\frac{\lambda - \mu - k + 1}{2} - 1\big) v, \\ \quad \text{Here } j = 1, \ldots, k, \quad E_\nu = \frac{\partial}{\partial t_\nu} \ (\nu = 1, \ldots, n). \end{cases}$$

on $A \ni \mathrm{diag}(e^{t_1}, \ldots, e^{t_n})$.

The dimension of the solution space equals 2^k.
There exists a unique solution with the moderate growth up to a constant multiple. It is written by the modified Bessel functions of the 2-nd kind.
(Similarly K-finite vectors $\Rightarrow$ expressed by Whittaker functions.)

Key to prove Theorem.

1. Irreducibility of a Whittaker module.

The left $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{C})$-module

$$\begin{cases} (E_{11} + E_{22} - \lambda - \mu)v = 0, \\ \Big(E_{11}E_{22} - E_{12}E_{21} + E_{11} - \lambda(\mu + 1)\Big)v = 0 \\ (E_{21} - c_1)v = 0 \end{cases}$$

is irreducible if $c_1 \neq 0$ (Kostant in general).

2. Twisted Harish-Chandra homomorphism.

$\mathfrak{g} = \bar{\mathfrak{n}} + \mathfrak{a} + \mathfrak{n}$: a complex reductive Lie algebra (or G is a normal real form)

$\gamma_\varpi : U(\mathfrak{g}) \to U(\mathfrak{a}),\ D \mapsto \gamma_\varpi(D)$ with

$$D - \gamma_\varpi(D) \in \bar{\mathfrak{n}}U(\mathfrak{g}) + U(\mathfrak{g}) \textstyle\sum_{X \in \mathfrak{n}} \Big(X - \varpi(X)\Big),$$

Remark. I: two-sided ideal of $U(\mathfrak{g})$ ($\Rightarrow \gamma_\varpi(I)$ is an ideal of $U(\mathfrak{a})$)

i) $\mathrm{supp}\,\varpi \subset \mathrm{supp}\,\varpi' \Rightarrow \gamma_\varpi(I) \subset \gamma_{\varpi'}(I)$

ii) If $\mathrm{gr}(I)$ does not vanish at $\sum_{\alpha \in \mathrm{supp}\,\varpi} X_\alpha$, then $\gamma_\varpi(I) = U(\mathfrak{a})$.

Characterize $\gamma_\varpi\Big(\mathrm{Ann}(\mathrm{Ind}_{P_\Theta}^G(\lambda)\Big)$!

3. A Boundary value problem on $K_{\mathbb{C}} \times AN$

A boundary attached to infinite points of $\exp(\sqrt{-1}\mathfrak{t} + \mathfrak{a})$ corresponding to a certain Weyl chamber. Here $\mathfrak{t}$ is a maximal torus of $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$.

4. Integral expression of Whittaker model with moderate growth

The kernel function of the intertwining operator is a distribution but its support has in general no inner point.
$W(\mathrm{supp}\varpi, \Theta)$ gives the possibility of the support.

That's all !
Thank you.