Subsystems of a Root System (E_8)

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§1. A classification of root systems

Definition. $\mathcal{G}$: *Affine Dynkin diagram*

A diagram consists of vertices and lines/arrows connecting vertices.

Each arrow has a stem with multiple lines.

Every vertex has a value of a positive real number.

The minimal real number in the connected component of $\mathcal{G}$ equals 1.

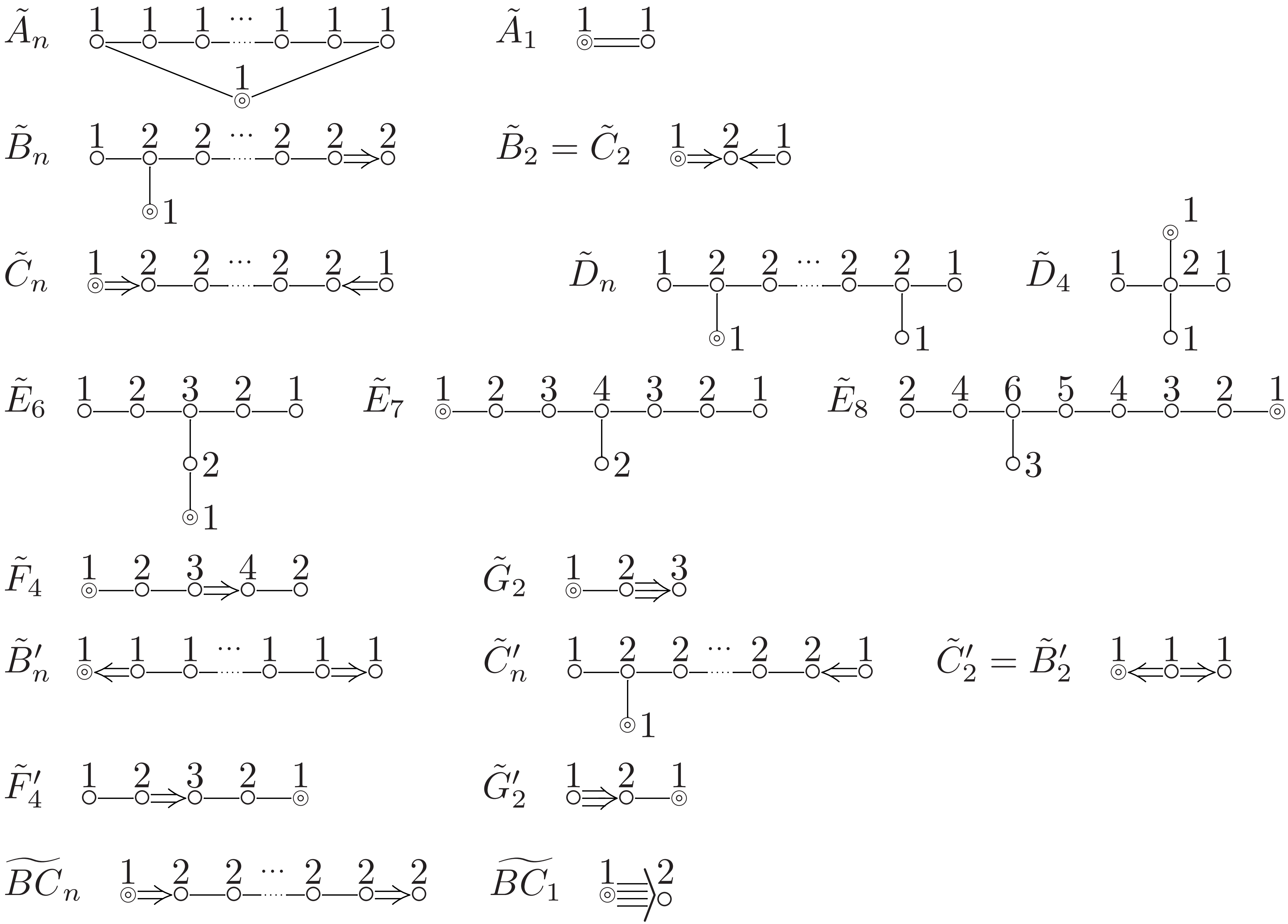
Fix any vertex P and suppose P has a value m.

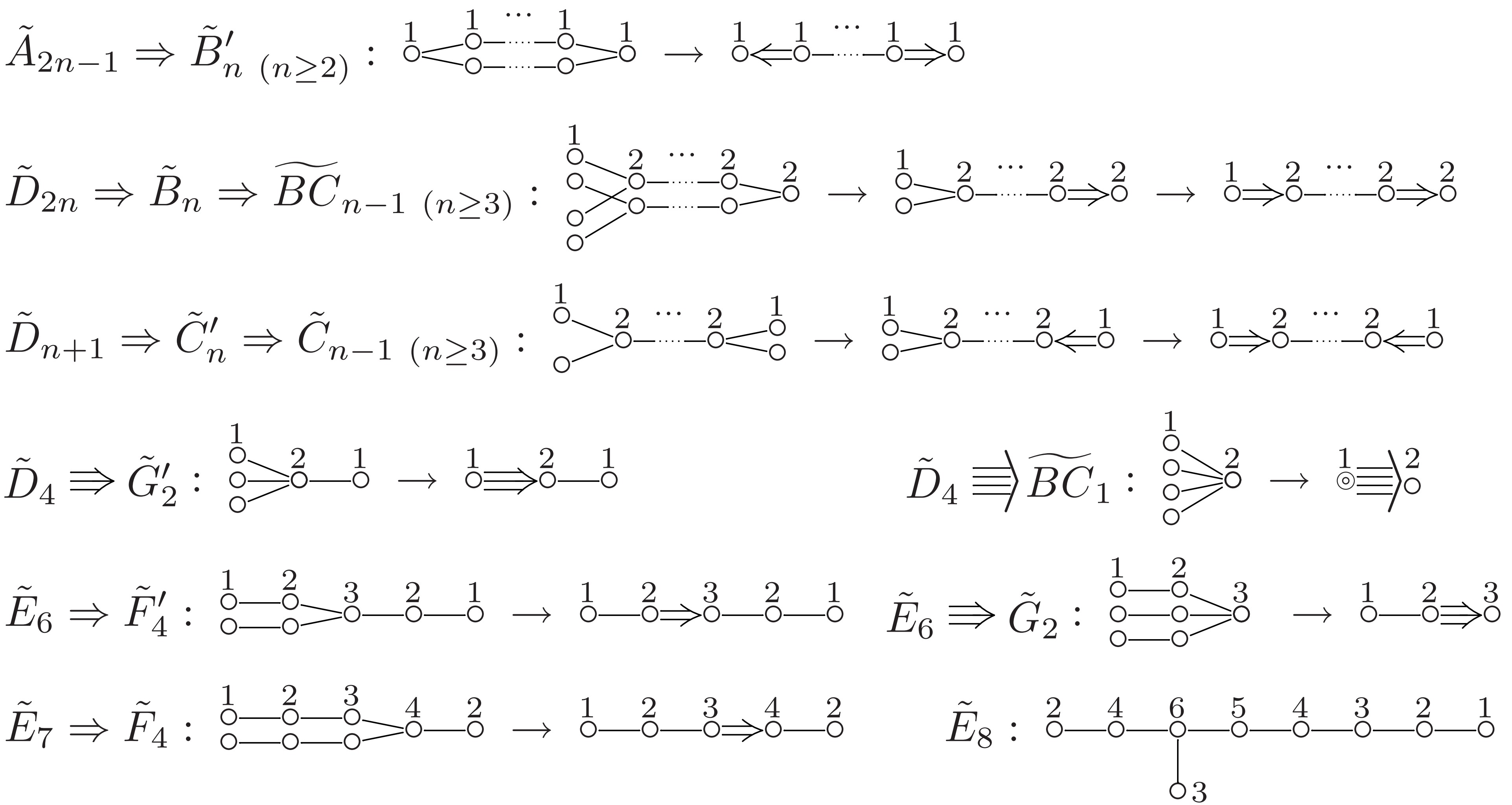
Let $L_1, \ldots, L_p$ be lines/arrows connecting P to vertices $Q_1, \ldots, Q_p$ with values $m_1, \ldots, m_p$, respectively. Then

$$2m = k_1 m_1 + \cdots + k_p m_p. \tag{1}$$

Here $k_j \neq 1 \Rightarrow L_j$ is an arrow with k_j-tuple lines pointing toward P.

Theorem 1. The connected affine Dynkin diagrams is $G(\tilde{\Psi})$, $G(\tilde{\Psi}')$ or $G(\widetilde{BC}_n)$ with the fundamental systems Ψ of an irreducible root system Σ (by adding one root in the closure of the negative Weyl chamber)

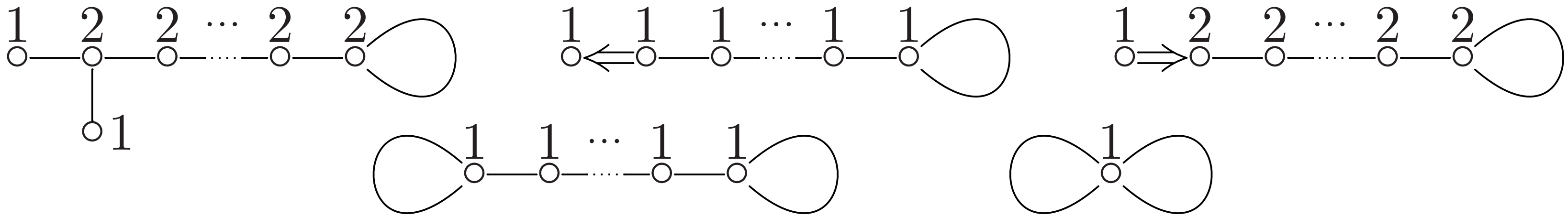

$\tilde{A}_n$
$\tilde{A}_1$
$\tilde{B}_n$
$\tilde{B}_2 = \tilde{C}_2$
$\tilde{C}_n$
$\tilde{D}_n$
$\tilde{D}_4$
$\tilde{E}_6$
$\tilde{E}_7$
$\tilde{E}_8$
$\tilde{F}_4$
$\tilde{G}_2$
$\tilde{B}'_n$
$\tilde{C}'_n$
$\tilde{C}'_2 = \tilde{B}'_2$
$\tilde{F}'_4$
$\tilde{G}'_2$
$\widetilde{BC}_n$
$\widetilde{BC}_1$



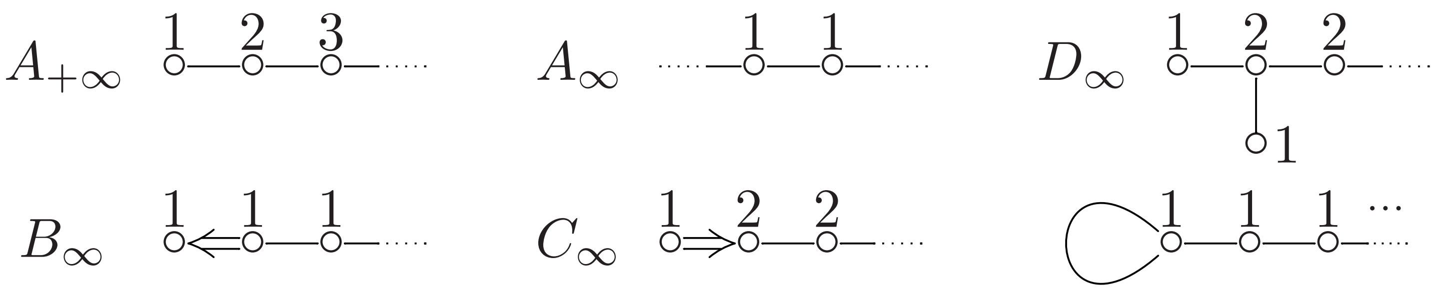
$$\tilde{R} = R^{(1)}\ \big(R = A_n,\ B_n,\ C_n,\ D_n,\ E_6,\ E_7,\ E_8,\ F_4,\ G_2\big),$$

$$\tilde{B}'_n = D^{(2)}_{n+1},\ \tilde{C}'_n = A^{(2)}_{2n-1},\ \widetilde{BC}_n = A^{(2)}_{2n},\ \tilde{F}'_4 = E^{(2)}_6,\ \tilde{G}'_2 = D^{(3)}_4.$$

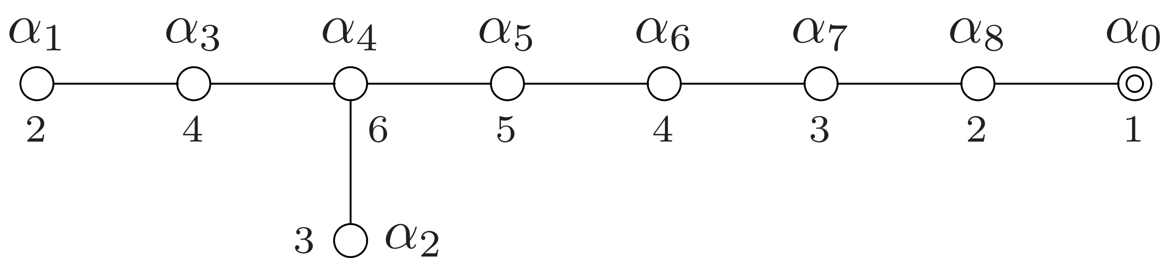
Allow lines/arrows connecting the same vertices



Allow infinite number of vertices



E_8



The numbers indicate the linear relation of the roots.

$\{\alpha_2, \alpha_3, \alpha_4, \ldots, \alpha_8, \alpha_0\}$: Type D_8

$$\alpha_2 = \epsilon_1 + \epsilon_2, \quad \alpha_j = \epsilon_{j-1} - \epsilon_{j-2} \quad (3 \le j \le 8),\ \alpha_0 = -\epsilon_7 - \epsilon_8,$$

This linear relation $\Rightarrow$

$$\begin{aligned} \alpha_1 &= -\tfrac{1}{2}(\alpha_0 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8) \\ &= \tfrac{1}{2}(\epsilon_1 + \epsilon_8) - \tfrac{1}{2}(\epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7), \end{aligned}$$

$$\Sigma = \Big\{\pm(\epsilon_i - \epsilon_j),\ \pm(\epsilon_i + \epsilon_j),\ \tfrac{1}{2}\textstyle\sum_{k=1}^{8}(-1)^{\nu(k)}\epsilon_k\,;\ 1 \le i < j \le 8,$$
$$\textstyle\sum_{k=1}^{8}\nu(k) \text{ is even}\Big\}, \quad \#\Sigma = 240.$$

§2. Subsystems of a root system

Σ: a (reduced) root system of rank n (a finite subset of $\subset \mathbb{R}^n \setminus \{0\}$)

i) $s_\alpha(\Sigma) = \Sigma \quad (\forall \alpha \in \Sigma)$ $\quad \left(s_\alpha(x) := x - 2\frac{(\alpha|x)}{(\alpha|\alpha)}\alpha, \quad x \in \mathbb{R}^n\right)$

ii) $2\frac{(\alpha|\beta)}{(\alpha|\alpha)} \in \mathbb{Z} \quad (\forall \alpha,\ \beta \in \Sigma)$

iii) $\sum_{\alpha\in\Sigma} \mathbb{R}\alpha = \mathbb{R}^n$ iv) $\Sigma \cap \mathbb{R}\alpha = \{\pm\alpha\} \quad (\forall \alpha \in \Sigma)$

$F \subset \Sigma$

$W_F := \langle s_\alpha; \alpha \in F\rangle \quad \subset O(n)$ $\quad W := W_\Sigma$: the Weyl group of Σ

$\langle F\rangle := W_F F$

F : a subsystem of $\Sigma \iff \langle F\rangle = F$

Ξ, Σ : root systems

$\mathrm{Hom}(\Xi,\Sigma) := \left\{\iota : \Xi \to \Sigma\,;\, 2\frac{(\iota(\alpha)|\iota(\beta))}{(\iota(\alpha)|(\iota(\alpha))} = 2\frac{(\alpha|\beta)}{(\alpha|\alpha)} \quad (\forall \alpha,\ \beta \in \Xi)\right\}$

$\overline{\mathrm{Hom}}(\Xi,\Sigma) := W_\Sigma \backslash \mathrm{Hom}(\Xi,\Sigma)$

$\mathrm{Aut}(\Sigma) := \mathrm{Hom}(\Sigma,\Sigma),\quad \mathrm{Out}(\Sigma) := \overline{\mathrm{Hom}}(\Sigma,\Sigma)$

$\Xi \simeq \Sigma$ (isomorphic) $\overset{\text{def}}{\iff} \exists \iota \in \mathrm{Hom}(\Xi,\Sigma)$ such that $\iota(\Xi) = \Sigma$

$\Xi,\ \Xi'$: subsystems of Σ

$\Xi \underset{\Sigma}{\sim} \Xi'$: (equivalent by Σ) $\exists w \in W_\Sigma$ such that $w(\Xi) = \Xi'$

$\Xi \overset{w}{\underset{\Sigma}{\sim}} \Xi'$: (weakly equivalent) $\exists g \in \mathrm{Aut}(\Sigma)$ such that $g(\Xi) = \Xi'$

Q1. For given root systems Σ and Σ', $\mathrm{Hom}(\Sigma', \Sigma) \neq \emptyset$?

Q2. $\Xi \simeq \Xi' \ \Rightarrow \Xi \underset{\Sigma}{\sim} \Xi'$? If not, how to distinguish them?

Q3. $\#\{F \subset \Sigma\,;\, F \underset{\Sigma}{\sim} \Xi\}$?

Q4. Does $\mathrm{Aut}(\Xi)$ come from W_Σ? (ex. orthogonal systems)

Q5. $\sigma \in \mathrm{Aut}(\Xi)$: stabilizes each irreducible component of Ξ

$\Rightarrow$ σ is realized by an element of W?

Q6. $\Xi' = \sigma(\Xi)$ with $\exists \sigma \in \mathrm{Out}(\Sigma) \ \Rightarrow \ \Xi \underset{\Sigma}{\sim} \Xi'$?

Q7. $\#\{\Theta \subset \Psi\,;\, \langle\Theta\rangle \underset{\Sigma}{\sim} \Xi\}$? ($\Psi$: a fundamental system of Σ)

Q8. $$\overline{\mathrm{Hom}}(\Xi, \Sigma),\ \mathrm{Out}(\Sigma)\backslash\overline{\mathrm{Hom}}(\Xi, \Sigma),$$
$$\overline{\mathrm{Hom}}(\Xi, \Sigma)/\mathrm{Out}(\Xi),\ \overline{\mathrm{Hom}}(\Xi, \Sigma)/\mathrm{Out}'(\Xi)$$

Q9. Closures of Ξ ($\perp$-, S-, L-, fundamental)?

$\mathrm{Aut}'(\Xi) := \mathrm{Aut}(\Xi'_1) \times \cdots \times \mathrm{Aut}(\Xi'_m) \subset \mathrm{Aut}(\Xi)$ $(\Xi = \Xi'_1 + \cdots + \Xi'_m)$

$\Xi^\perp := \{\alpha \in \Sigma\,;\, \alpha \perp \Xi\}$

Ξ : $\perp$-closed $\overset{\text{def}}{\Leftrightarrow}$ $\Xi = (\Xi^\perp)^\perp$

Ξ : L-closed $\overset{\text{def}}{\Leftrightarrow}$ $\sum_{\alpha \in \Xi} \mathbb{R}\alpha \cap \Sigma = \Xi$

Ξ : S-closed $\overset{\text{def}}{\Leftrightarrow}$ $\alpha \in \Xi$, $\beta \in \Xi$, $\alpha + \beta \in \Sigma \Rightarrow \alpha + \beta \in \Xi$

Ξ : fundamental $\overset{\text{def}}{\Leftrightarrow}$ $\exists$ a fundamental system of Σ containing Ξ

$\perp$-closed $\Rightarrow$ fundamental $\Leftrightarrow$ L-closed $\Rightarrow$ S-closed

Q10. $\mathrm{Out}_\Sigma(\Xi) := \{w|_\Xi\,;\, w \in W_\Sigma$ with $w(\Xi) = \Xi\} \,/\, W_\Xi$?

Dynkin, *"Semisimple subalgebras of semisimple Lie algebras"*, 1957.

Aslaksen and Lang, *"Extending π-systems to bases of root systems"*, 2005.

§3. $\overline{\mathrm{Hom}}(\Xi, \Sigma)$

Lemma. $$\overline{\mathrm{Hom}}(\Xi_1 + \Xi_2, \Sigma) \simeq \coprod_{\bar\iota \in \overline{\mathrm{Hom}}(\Xi_1, \Sigma)} \Big(\bar\iota, \overline{\mathrm{Hom}}\big(\Xi_2, \iota(\Xi_1)^\perp\big)\Big)$$

Lemma $\Rightarrow$ Study $\iota \in \overline{\mathrm{Hom}}(\Xi, \Sigma)$ and $\iota(\Xi)^\perp$ for irreducible Ξ and Σ.

Theorem 2. i) Σ : classical type or $\Xi \simeq A_m$ with $m \geq 1$.

When $\Xi \not\simeq D_4$ or $(\Xi, \Sigma) \simeq (D_4, D_4)$,

$$\overline{\mathrm{Hom}}(\Xi, \Sigma) \xleftarrow{\sim} \{\text{Imbeddings } \bar{\iota} \text{ of } G(\Phi) \text{ to } G(\tilde{\Psi}) \text{ or } G(\tilde{\Psi}') \text{ such that } \bar{\iota}(\beta_0) = \alpha_0 \text{ or } \alpha_0'\} \tag{2}$$

β_0 : any root in Φ such that the right hand side of (2) is not empty and if such β_0 doesn't exit, $\overline{\mathrm{Hom}}(\Xi, \Sigma) = \emptyset$.

When $\Xi = D_4$, $\quad \#\big(\overline{\mathrm{Hom}}(D_4, \Sigma)/\mathrm{Out}(D_4)\big) \leq 1 \qquad (3)$

and $\overline{\mathrm{Hom}}(D_4, \Sigma)/\mathrm{Out}(D_4)$ is given by the above $\bar{\iota}$

$$\#\overline{\mathrm{Hom}}(D_4, B_n) = \#\overline{\mathrm{Hom}}(D_4, C_n) = \#\overline{\mathrm{Hom}}(D_4, D_{n+1}) = 3 \quad (n \geq 4)$$

In general

$$\iota(\Xi)^{\perp} \simeq \langle \alpha \in \Psi \,;\, \alpha \perp \bar{\iota}(\Phi) \rangle \tag{4}$$

ii) Σ : exceptional type and $\Xi = R_m (= B_m,\ C_m,\ D_m,\ E_m,\ F_4$ or $G_2)$

$m \geq m_0^R := 2,\ 3,\ 4,\ 6,\ 4,\ 2$ ($R = B,\ C,\ D,\ E,\ F,\ G$, respectively)

m_Σ^R : maximal m such that $G(R_m) \subset G(\tilde{\Psi})$ or $G(\tilde{\Psi}')$

$G(R_{m_\Sigma^R}) \simeq G(\Phi_\Sigma^R)$ with $\Phi_\Sigma^R \subset \tilde{\Psi}$ or $\tilde{\Psi}'$ ($m_\Sigma^R = 0$ if it doesn't exist)

When $(R_m, \Sigma) \not\simeq (D_4, F_4)$,

$$\#\big(\overline{\mathrm{Hom}}(R_m, \Sigma)/\mathrm{Out}(R_m)\big) \leq 1, \tag{2}$$

$$\#\overline{\mathrm{Hom}}(R_m, \Sigma) = \begin{cases} 0 & (m > m_\Sigma^R), \\ \#\,\mathrm{Out}(R_{m_\Sigma^R}) & (m = m_\Sigma^R), \\ 1 & (m_0^R \leq m < m_\Sigma^R), \end{cases} \tag{5}$$

$R_m^\perp \cap \Sigma = (R_m^\perp \cap R_{m_\Sigma^R}) + \big\langle (\Phi_\Sigma^R)^\perp \cap \tilde{\Psi}$ (or $\tilde{\Psi}'$)$\big\rangle$ $(m_0^R \leq m \leq m_\Sigma^R)$

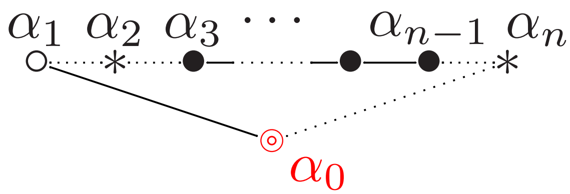
through $G(R_m) \subset G(R_{m_\Sigma^R}) \simeq G(\Phi_\Sigma^R) \subset G(\tilde{\Psi})$ (or $G(\tilde{\Psi}')$)

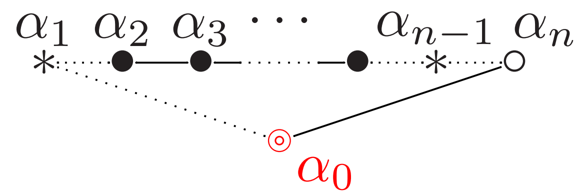
and $R_m^\perp \cap R_{m_\Sigma^R}$ is given by i) or (4).

$\#\overline{\mathrm{Hom}}(D_4, F_4) = 2 \ \Leftarrow\ D_4 \simeq F_4^L$ and $D_4 \simeq F_4 \setminus F_4^L$.

Examples

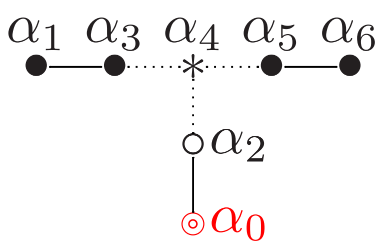
i) $\#\overline{\mathrm{Hom}}(A_2, A_n) = 2$ and $A_2^\perp \cap A_n \simeq A_{n-3}$ $(n \geq 2)$.

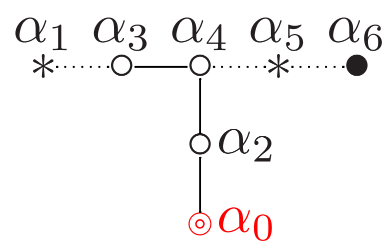


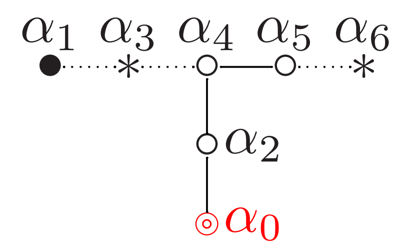


ii) $\#\overline{\mathrm{Hom}}(A_2, E_6) = 1$ and $A_2^\perp \cap E_6 \simeq 2A_2$, $\quad \#\overline{\mathrm{Hom}}(3A_2, E_6) = 8$.

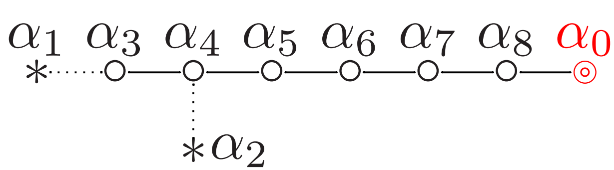
$\#\overline{\mathrm{Hom}}(A_4, E_6) = 2$, $\#\big(\mathrm{Out}(E_6)\backslash\overline{\mathrm{Hom}}(A_4, E_6)\big) = 1$ and $A_4^\perp \cap E_6 \simeq A_1$.

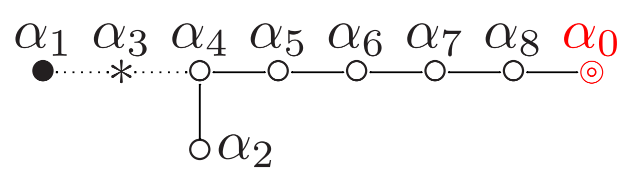






iii) $\#\overline{\mathrm{Hom}}(A_7, E_8) = \#\big(\mathrm{Out}(E_8)\backslash\overline{\mathrm{Hom}}(A_7, E_8)/\mathrm{Out}(A_7)\big) = 2$.





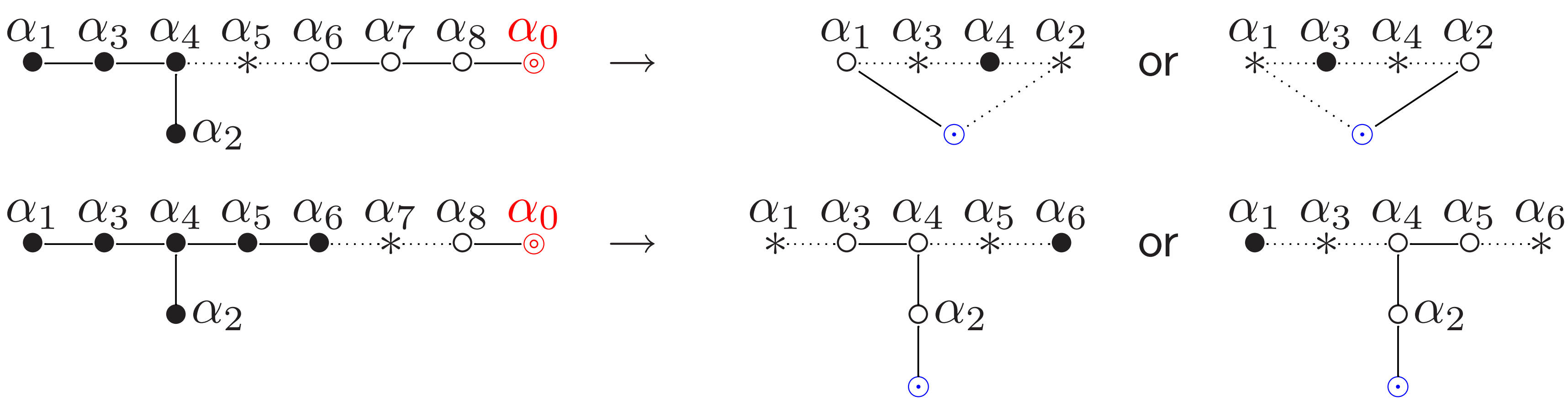
$A_7^\perp \cap E_8 = \emptyset$ $\qquad$ $A_7^\perp \cap E_8 \simeq A_1$

$\#\overline{\mathrm{Hom}}$:

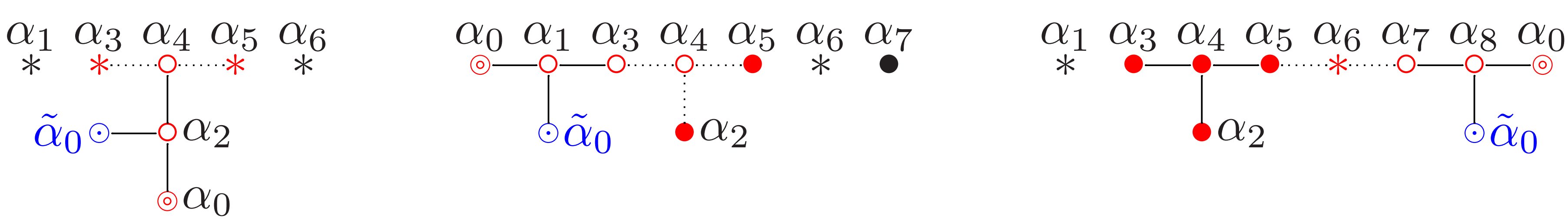
$\Sigma \setminus \Xi$	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
E_6	1	1	1	2(1)	2(1)	0	0	0
E_7	1	1	1	1	2(2)	1	1	0
E_8	1	1	1	1	1	1	2(2)	1

iv) $\#\overline{\mathrm{Hom}}(A_4 + A_2, E_8) = 2$ and $\#\big(\overline{\mathrm{Hom}}(A_4 + A_2, E_8)/\mathrm{Out}(A_4 + A_2)\big) = 1$.

$(A_4 + A_2)^{\perp} \cap E_8 \simeq A_1$.

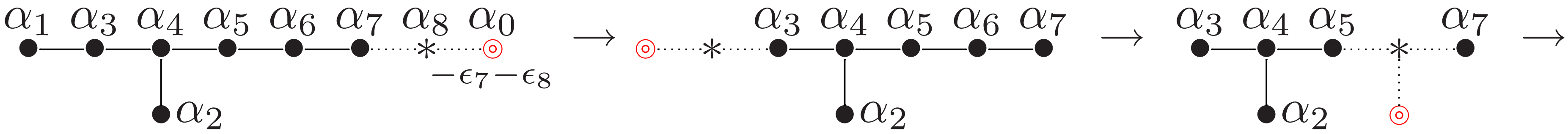


v) $\#\overline{\mathrm{Hom}}(D_4, E_n) = 1$

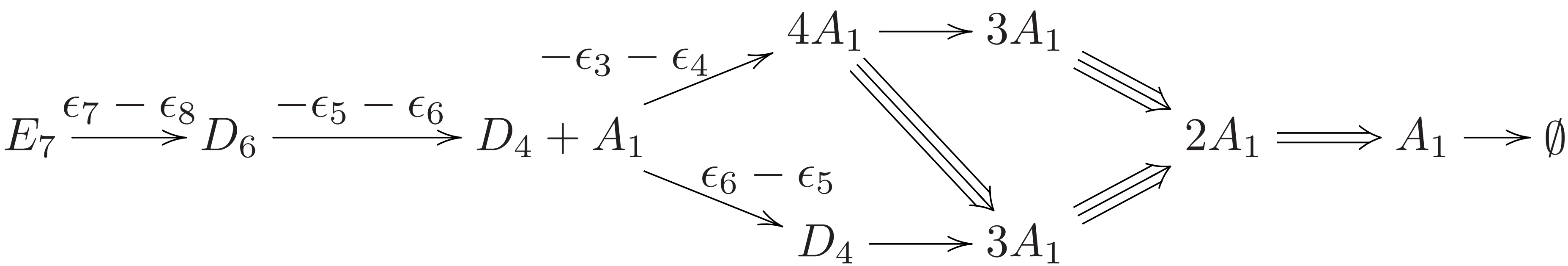


$D_4^{\perp} \cap E_6 = \emptyset$ $\quad$ $D_4^{\perp} \cap E_7 \simeq 3A_1$ $\quad$ $D_4^{\perp} \cap E_8 \simeq D_4$

vi) $\overline{\mathrm{Hom}}(mA_1, E_8)$, $\overline{\mathrm{Hom}}(mA_1, E_7)$



$$E_8 \xrightarrow{-\epsilon_7 - \epsilon_8} E_7$$



$$\#\overline{\mathrm{Hom}}(kA_1, E_8) = \#\overline{\mathrm{Hom}}((k-1)A_1, E_7) = \begin{cases} 1 & (1 \le k \le 3) \\ 2 & (k = 4) \\ 5 & (k = 5) \\ 15 & (k = 6) \\ 30 & (7 \le k \le 8) \\ 0 & (k > 8) \end{cases}$$

$\mathrm{Out}_{D_8}(8A_1) \subset \mathrm{Out}_{E_8}(8A_1) \subset \mathrm{Out}(8A_1) \simeq \mathfrak{S}_8$, $\quad \#\,\mathrm{Out}_{E_8}(8A_1) = \frac{8!}{30} = 1344$

§4. Duality

(Ξ_1, Ξ_2) : a dual pair in $\Sigma \overset{\text{def}}{\Leftrightarrow} \Xi_1, \Xi_2 \subset \Sigma,\ \Xi_1 = \Xi_2^{\perp},\ \Xi_2 = \Xi_1^{\perp}$

A dual pair (Ξ_1, Ξ_2) of Σ is special $\overset{\text{def}}{\Leftrightarrow}$

$$\mathrm{Out}(\Xi_1) \simeq \{w|_{\Xi_1}\,;\, w \in W_\Sigma,\ w(\Xi_1) = \Xi_1\}/W_{\Xi_1} \simeq \mathrm{Out}(\Xi_2)$$

Theorem 3. A dual pair (Ξ_1, Ξ_2) in Σ is special

$\Leftarrow$ $\#\overline{\mathrm{Hom}}(\Xi_1, \Sigma) = 1$ and $\mathrm{Out}(\Xi_1) \simeq \mathrm{Out}(\Xi_2)$

$\Leftarrow$ $\#\overline{\mathrm{Hom}}(\Xi_1, \Sigma) = \#\overline{\mathrm{Hom}}(\Xi_2, \Sigma) = 1$

Example. (Σ, Ξ_1, Ξ_2) : (Ξ_1, Ξ_2) are special dual pairs in Σ.

$$(D_{m+n}, D_m, D_n) \quad (m \geq 2,\ n \geq 2,\ m \neq 4,\ n \neq 4),$$
$$(E_6, A_3, 2A_1),\ (E_7, A_5, A_2),\ (E_7, A_3 + A_1, A_3),\ (E_7, 3A_1, D_4),$$
$$(E_8, E_6, A_2),\ (E_8, A_5, A_2 + A_1),\ (E_8, A_4, A_4),\ (E_8, D_6, 2A_1),$$
$$(E_8, D_5, A_3),\ (E_8, D_4, D_4),\ (E_8, D_4 + A_1, 3A_1),\ (E_8, 2A_2, 2A_2),$$
$$(E_8, A_3 + A_1, A_3 + A_1),\ (E_8, 4A_1, 4A_1),\ (F_4, A_2, A_2).$$

Σ is $E_8 \Rightarrow$ all the dual pairs are special!

Result. Using Theorem 2 and 3, we get Tables which answer Q1–Q10.

“A classification of subsystems of a root system”, 47pp.

`http://akagi.ms.u-tokyo.ac.jp/~oshima/` or `math.RT/0611904`

$\Xi \quad \backslash \quad \Sigma$	E_6	E_7	E_8	F_4	G_2
equivalent (isomorphic)	20 (20)	46 (40)	76 (71)	36 (22)	6 (4)
S-closed	20	46	76	23	5
L-closed ($\perp$-closed)	16 (7)	31 (13)	40 (18)	11 (9)	3 (3)
$\Xi^{\perp} = \emptyset$ (rank Ξ = rank Σ)	10 (3)	19 (7)	33 (13)	20 (16)	4 (4)
maximal (S-closed)	3 (3)	4 (4)	5 (5)	3 (3)	3 (2)
dual pairs (special)	3 (1)	6 (3)	11 (11)	5 (4)	1 (1)