# ON CONVERGENCE OF BASIC HYPERGEOMETRIC SERIES

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Abstract. We examine the convergence of $q$-hypergeometric series when $|q| = 1$. We give a condition so that the radius of the convergence is positive and get the radius. We also show that the numbers $q$ with the positive radius of the convergence are densely distributed in the unit circle of the complex plane of $q$ and so are those with the radius 0.

## 1. Introduction

Basic hypergeometric series (cf. [GR]) with the base $q$ is defined by

$$ {}_r\phi_s\left[\begin{matrix}a_1,a_2,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};q,z\right]=\sum_{n=0}^{\infty}\frac{(a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n}{(q;q)_n(b_1;q)_n\cdots(b_s;q)_n}\bigl((-1)^nq^{\frac{n(n-1)}{2}}\bigr)^{s+1-r}z^n, $$

where

$$ (a;q)_n=\prod_{j=1}^{n}(1-aq^{j-1}) $$

is the $q$-Pochhammer symbol. Here $a_1,\ldots,a_r$, $b_1,\ldots,b_s$ and $q$ are complex parameters. In this paper we always assume

$$ a_iq^n\neq 1 \text{ and } b_jq^n\neq 1 \qquad (i=1,\ldots,r,\ j=1,\ldots,s,\ n=0,1,2,\ldots) \tag{1} $$

so that the factors $(a_i;q)_n$ and $(b_j;q)_n$ in the terms of the series are never zero.

Let $v_n$ be the terms of the series ${}_r\phi_s$ which contain $z^n$. Then we have

$$ \begin{aligned}\frac{v_{n+1}}{v_n}&=\frac{(1-a_1q^n)(1-a_2q^n)\cdots(1-a_rq^n)}{(1-q^{n+1})(1-b_1q^n)\cdots(1-b_sq^n)}(-q^n)^{1+s-r}z\\ &=\frac{(a_1-q^{-n})(a_2-q^{-n})\cdots(a_r-q^{-n})}{(1-q^{-n-1})(b_1-q^{-n})\cdots(b_s-q^{-n})}\frac{z}{q}.\end{aligned} $$

If $0<|q|<1$, the radius of convergence of the series ${}_r\phi_s$ equals $\infty$ if $r\le s$ and equals 1 if $r=s+1$. If $|q|>1$ and

$$ a_1\cdots a_rb_1\cdots b_s\neq 0, \tag{2} $$

the radius of convergence of the series equals

$$ \frac{|b_1b_2\cdots b_sq|}{|a_1a_2\cdots a_r|}. \tag{3} $$

In this paper we discuss the convergence of the series when $|q|=1$. The convergence of ${}_2\phi_1$ is assumed in [OS] but it is a subtle problem depending on the base

$$ q=e^{2\pi i\theta}. \tag{4} $$

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We assume that $\theta$ is not a rational number, namely, $\theta \in \mathbb{R} \setminus \mathbb{Q}$ so that $(q;q)_n$ never vanish and we have the following theorem.

**Theorem 1.** *Retain the notation above and assume the conditions* (1) *and* (2).

i) *Assume that there exists a positive number $C$ such that*

$$\left|\theta - \frac{k}{m}\right| > \frac{C}{m^2} \qquad (\forall k \in \mathbb{Z},\ m = 1, 2, 3, \ldots).$$

*Then we have*

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(e^{2\pi i\theta}; e^{2\pi i\theta}\right)_n\right|} = 1. \tag{5}$$

*Suppose moreover that every parameter $a_i$ or $b_j$ has an absolute value different from* 1 *or equals $e^{2\pi i\alpha} q^{\beta}$ with suitable rational numbers $\alpha$ and $\beta$ which may depend on $a_i$ and $b_j$. Then the radius of convergence of the series ${}_r\phi_s$ equals*

$$\frac{\max\{|b_1|, 1\} \cdots \max\{|b_s|, 1\}}{\max\{|a_1|, 1\} \cdots \max\{|a_r|, 1\}}. \tag{6}$$

ii) *In general, we have*

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\left|(e^{2\pi i\theta}; e^{2\pi i\theta})_n\right|} \le 1 \qquad (\forall \theta \in \mathbb{R} \setminus \mathbb{Q}). \tag{7}$$

*The set of irrational real numbers $\theta$ satisfying*

$$\underline{\lim_{n\to\infty}} \sqrt[n]{\left|(e^{2\pi i\theta}; e^{2\pi i\theta})_n\right|} = 0 \tag{8}$$

*is dense in $\mathbb{R}$ and uncountable. If $\theta$ satisfies* (8) *and the absolute value of any parameter $a_i$ or $b_j$ is not* 1*, the radius of convergence of the series ${}_r\phi_s$ equals* 0.

Note that it is known that an irrational number $\theta$ satisfies the assumption in Theorem 1 i) if and only if the positive integers appearing in its expansion of continued fraction are bounded and hence the set of real numbers $\theta$ satisfying it is uncountable and dense in $\mathbb{R}$.

Suppose $\theta \in \mathbb{R} \setminus \mathbb{Q}$ satisfies the assumption in Theorem 1 i). Then

$$\left|\frac{k_1}{m_1}\theta - \frac{k_2}{m_2} - \frac{k}{m}\right| = \left|\frac{k_1}{m_1}\right| \cdot \left|\theta - \frac{m_1(k_2 m + k m_2)}{k_1 m_2 m}\right| > \frac{C}{k_1 m_1 m_2^2 \cdot m^2} \tag{9}$$

for integers $k, k_1, k_2, m, m_1, m_2$ with $k_1 m m_1 m_2 \neq 0$ and therefore the number $r_1\theta + r_2$ with $r_1 \in \mathbb{Q} \setminus \{0\}$ and $r_2 \in \mathbb{Q}$ also satisfies the assumption.

**Example 2.** The estimate

$$\left|\sqrt{2} - \frac{k}{m}\right| > \frac{1}{3m^2} \qquad (\forall k \in \mathbb{Z},\ m = 1, 2, 3, \ldots) \tag{10}$$

shows that the real number $\theta = \sqrt{2} + r$ with $r \in \mathbb{Q}$ satisfies the assumption in Theorem 1 i).

We will prove (10). We assume the existence of integers $k$ and $m$ satisfying $m \ge 1$ and $|\sqrt{2} - \frac{k}{m}| \le \frac{1}{3m^2}$. Then we may moreover assume $m \ge 2$ and therefore

$$\begin{aligned}
1 \le \left|2m^2 - k^2\right| &= \left|\left(\sqrt{2}m - k\right)\left(\sqrt{2}m + k\right)\right| \\
&= \left|m^2\left(\sqrt{2} - \frac{k}{m}\right)\right| \cdot \left|2\sqrt{2} - \left(\sqrt{2} - \frac{k}{m}\right)\right| \\
&\le \frac{1}{3}\left(2\sqrt{2} + \frac{1}{3m^2}\right) \le \frac{2\sqrt{2}}{3} + \frac{1}{9 \cdot 4} = 0.9705 \cdots < 1,
\end{aligned}$$

which leads a contradiction.

We will show Theorem 1 i) in §4 and Theorem 1 ii) in §5.

## 2. Preliminary results

First we review the following theorem which claims that $k\theta \mod \mathbb{Z}$ for $k = 1, 2, \ldots$ are uniformly distributed on $\mathbb{R}/\mathbb{Z}$.

**Theorem 3** (Bohl, Sierpiński and Weil)**.** *Let $f(x)$ be a periodic function on $\mathbb{R}$ with period 1. If $f(x)$ is integrable in the sense of Riemann, then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(k\theta) = \int_0^1 f(x)\,dx \qquad (\forall\theta\in\mathbb{R}\setminus\mathbb{Q}). \tag{11}$$

This theorem is proved by approximating $f(x)$ by a finite Fourier series (cf. [AA]) since the theorem is directly proved if $f(x)$ is a finite Fourier series with the fact

$$\sum_{k=1}^{n}\frac{e^{2\pi imk\theta}}{n} = \frac{e^{2\pi im\theta}}{n}\left(\frac{1-e^{2\pi imn\theta}}{1-e^{2\pi im\theta}}\right) \xrightarrow{n\to\infty} 0 \qquad (m\neq 0).$$

We also prepare the following integral formula.

$$\int_0^1 \log\left|1-re^{2\pi ix}\right| dx = \begin{cases} 0 & (0\le r\le 1),\\ \log r & (r\ge 1).\end{cases} \tag{12}$$

The series $-\frac{\log(1-z)}{z} = 1 + \frac{z}{2} + \frac{z^2}{3} + \cdots$ converges when $|z| < 1$ and therefore

$$0 = \int_{|z|=r}\frac{\log(1-z)}{z}\,dz = 2\pi i\int_0^1 \log(1-re^{2\pi ix})\,dx \qquad (0\le r<1,\ z=2\pi ix).$$

by Cauchy's integral formula. Since

$$\mathrm{Re}\int_0^1 \log(1-re^{2\pi ix})\,dx = \int_0^1 \mathrm{Re}\log(1-re^{2\pi ix})\,dx = \int_0^1 \log|1-re^{2\pi ix}|\,dx,$$

we have (12) when $0 \le r < 1$. Moreover the relation

$$\log|1-re^{2\pi ix}| = \log r + \log|r^{-1}-e^{2\pi ix}| = \log r + \log|1-r^{-1}e^{2\pi ix}|$$

assures (12) when $r > 1$.

Note that the expansion $e^\sigma - 1 = \sigma(1+\frac{\sigma}{2!}+\frac{\sigma^2}{3!}+\cdots)$ assures $|1-e^\sigma| \ge \frac{|\sigma|}{2}$ when $|\sigma| < 1$. Hence if $r = 1$, the improper integration in (12) converges because

$$\left|\log|1-e^{2\pi iz}|\right| > \left|\log|\pi z|\right| \quad \text{for} \quad 0<|2\pi z|<1 \tag{13}$$

and we obtain (12) by taking the limit $r \to 1-0$ (cf. [Ah, 5.3.5]).

## 3. A lemma

We prepare a lemma to prove Theorem 1 i).

**Lemma 4.** *Let $f(x)$ be a periodic function on $\mathbb{R}$ with period 1. Suppose that $f(x)$ is continuous on $[0,1]$ except for finite points $c_1,\ldots,c_p\in[0,1)$. Suppose there exist $r_j\in\mathbb{Q}$ for $j=1,\ldots,p$ such that*

$$c_j - r_j\theta\in\mathbb{Q} \quad \text{and} \quad (r_j+k)\theta - c_j\notin\mathbb{Z} \quad \text{for} \quad k=1,2,\ldots. \tag{14}$$

*Suppose moreover that there exist a positive number $\epsilon$ and a continuous function $h(t)$ on $(0,1]$ such that*

$$\begin{gathered} |f(x)| < h(|x-c_j|) \quad \text{for} \quad 0<|x-c_j|<\epsilon,\\ \int_0^1 h(t)\,dt<\infty \quad \text{and} \quad h(t_1)\ge h(t_2)\ge 0 \quad \text{if} \quad 0<t_1<t_2\le 1. \end{gathered} \tag{15}$$

*If $\theta\in\mathbb{R}\setminus\mathbb{Q}$ satisfies the assumption in Theorem 1 i), then (11) is valid. Here we note that the condition (15) assures that the improper integral in (11) converges.*

*Proof.* Put

$$J(j,n,\epsilon)=\bigl\{k\mid 1\le k\le n,\ \min_{m\in\mathbb{Z}}\{|k\theta-c_j-m|\}<\epsilon\bigr\}$$

and

$$I_\epsilon=\bigl\{x\in[0,1]\mid \min_{m\in\mathbb{Z}}|x-c_j-m|\ge\epsilon\ \text{ for }\ j=1,\dots,p\bigr\}.$$

Then Theorem 3 shows that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\substack{k\notin J(1,n,\epsilon)\cup\cdots\cup J(p,m,\epsilon)\\ 1\le k\le n}} f(k\theta)=\int_{I_\epsilon}f(x)\,dx$$

and therefore we have only to show

$$\lim_{\epsilon\to+0}\lim_{n\to\infty}\frac{1}{n}\sum_{k\in J(j,n,\epsilon)}|f(k\theta)|=0 \tag{16}$$

to get this lemma.

Fix $j$. Since Theorem 3 shows that

$$\lim_{n\to\infty}\frac{1}{n}\#J(j,n,\epsilon)=2\epsilon,$$

there exists a positive integer $N_\epsilon$ such that

$$\#J(j,n,\epsilon)\le 3\epsilon n\quad(\forall n\ge N_\epsilon).$$

Put

$$J(j,\epsilon,n)=\{k_1,k_2,\dots,k_L\}$$

with $L=\#J(j,\epsilon,n)$ so that

$$\min_{m\in\mathbb{Z}}|k_\nu\theta-c_j-m|\le\min_{m\in\mathbb{Z}}|k_{\nu'}\theta-c_j-m|\ \text{ if }\ 1\le\nu<\nu'\le L.$$

Note that $c_j=\frac{k_1}{m_1}\theta+\frac{k_2}{m_2}$ with integers $k_1,k_2,m_1,m_2$. In view of (9), we have

$$\left|k\theta-\frac{k_1}{m_1}\theta-\frac{k_2}{m_2}-m\right|=\left|\frac{m_1k-k_1}{m_1}\theta-\frac{k_2+mm_2}{m_2}\right|>\frac{C}{m_1m_2^2|mk-k_1|}$$

for $k=1,2,3,\dots$ satisfying $mk\ne k_1$. If $\frac{k_1}{m_1}$ is a positive integer, the assumption implies $\frac{k_2}{m_2}\notin\mathbb{Z}$. Hence replacing $C$ by a small positive number if necessary, we may assume

$$\min_{m\in\mathbb{Z}}|k\theta-c_j-m|>\frac{C}{k}\quad(k=1,2,\dots,\ j=0,1,\dots,p),$$

where we put $c_0=0$. In particular, we have

$$\min_{m\in\mathbb{Z}}|k\theta-k'\theta-m|>\frac{C}{k'-k}\ge\frac{C}{n}\quad(0\le k<k'\le n).$$

Thus we have

$$\min_{m\in\mathbb{Z}}|k_\nu\theta-c_j-m|>\frac{C\nu}{2n}\quad(1\le\nu\le L)$$

and

$$|f(k_\nu\theta)|<h\bigl(\tfrac{C\nu}{2n}\bigr)\quad(1\le\nu\le L<3\epsilon n).$$

Hence if $n\ge N_\epsilon$, we have

$$\begin{aligned}\frac{1}{n}\sum_{k\in J(j,n,\epsilon)}|f(k\theta)|&=\frac{1}{n}\sum_{\nu=1}^{L}|f(k_\nu\theta)|\le\frac{1}{n}\sum_{\nu=1}^{[3\epsilon n]}h(\tfrac{C\nu}{2n})\\&\le\int_0^{\frac{[3\epsilon n]}{n}}h\bigl(\tfrac{Cx}{2}\bigr)\,dx\le\frac{2}{C}\int_0^{\frac{6\epsilon}{C}}h(t)\,dt,\end{aligned}$$

which implies (16). Here $[3\epsilon n]$ denotes the largest integer satisfying $[3\epsilon n] \leq 3\epsilon n$. $\square$

**Problem 5.** *Let $f(x)$ be a function satisfying the assumption in Lemma 4. Is the equality* (11) *for $\theta \in \mathbb{R}$ valid almost everywhere in the sense of Lebesgue measure? Here we may assume $p = 1$, $c_1 = 0$ and $h(t) = |\log t|$ for our problem. Is it also valid without assuming the condition* (14)*?*

## 4. Estimate I

Let $a = re^{2\pi i\tau}$ and $q = e^{2\pi i\theta}$ with $\tau$, $\theta \in \mathbb{R}$ and $r > 0$. Then

$$\sqrt[n]{\big|(a;q)_n\big|} = \exp\left(\frac{1}{n}\sum_{k=0}^{n-1}\log\big|1 - re^{2\pi i(k\theta+\tau)}\big|\right). \tag{17}$$

If $r \neq 1$, Theorem 3 and (12) imply

$$\lim_{n\to\infty}\sqrt[n]{\big|(a;q)_n\big|} = \max\{|a|, 1\} \qquad (|a| \neq 1,\ q = e^{2\pi i\theta} \ \text{ with } \ \theta \in \mathbb{R}\setminus\mathbb{Q}). \tag{18}$$

Assume $r = 1$. Since

$$\sum_{\substack{\min_{m\in\mathbb{Z}}\{|k\theta - m|\} < \epsilon \\ 1\leq k\leq n}} \log\big|1 - e^{2\pi ik\theta}\big| \leq 0$$

and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\substack{\min_{m\in\mathbb{Z}}\{|k\theta - m|\} \geq \epsilon \\ 1\leq k\leq n}} \log\big|1 - e^{2\pi ik\theta}\big| = \int_{\epsilon}^{1-\epsilon}\log|1 - e^{2\pi ix}|\,dx$$

for any small positive number $\epsilon$, we have (7) in view of (12) and (17).

Now assume moreover that $\theta$ satisfies the assumption in Theorem 1 i). Then Lemma 4 with $f(x) = \log|1 - e^{2\pi ix}|$ and $h(t) = |\log \pi t|$ (cf. (13)) proves (5).

Suppose $\tau = \frac{k_1}{m_1}\theta + \frac{k_2}{m_2}$ with integers $k_1, k_2, m_1, m_2$ with $m_1 > 0$, $m_2 > 0$ and

$$\left(\frac{k_1}{m_1} + k - 1\right)\theta + \frac{k_2}{m_2} \notin \mathbb{Z} \qquad (k = 1, 2, 3, \ldots)$$

corresponding to (1), (2) and (14). Lemma 4 with $f(x) = \log\big|1 - e^{2\pi i(x + (\frac{k_1}{m_1} - 1)\theta)}\big|$, $h(t) = |\log \pi t|$ and $c_j = -(\frac{k_1}{m_1} - 1)\theta - \frac{k_2}{m_2}$ implies

$$\lim_{n\to\infty}\sqrt[n]{\big|\big(e^{2\pi i\tau}; e^{2\pi i\theta}\big)_n\big|} = 1. \tag{19}$$

Thus we have Theorem 1 i) by the estimates (18), (5) and (19).

## 5. Estimate II

Define a series of rapidly increasing positive integers $\{a_n\}$ by

$$a_1 = 2, \quad a_{n+1} = k_{n+1}\cdot a_n \cdot a_n! \qquad (k_{n+1} = 2 \text{ or } 3,\ n = 1, 2, 3, \ldots) \tag{20}$$

and put $\theta = \displaystyle\sum_{n=1}^{\infty}\frac{1}{a_n}$ and $q = e^{2\pi i\theta}$. Then $\theta \notin \mathbb{Q}$ and we have

$$\min_{m\in\mathbb{Z}}\big|a_n\cdot\theta - m\big| < \sum_{k=n+1}^{\infty}\frac{a_n}{a_k} < \frac{1}{a_n!}, \tag{21}$$

$$\big|1 - e^{2\pi ia_n\theta}\big| \leq \frac{2\pi}{a_n!}, \quad \prod_{j=1}^{a_n}\big|1 - q^j\big| \leq \frac{2^{a_n}\pi}{a_n!}, \quad \lim_{n\to\infty}\sqrt[a_n]{\prod_{j=1}^{a_n}\big|1 - q^j\big|} = 0$$

and (8) in Theorem 1 ii). We may choose $k_n \in \{2,3\}$ for $n = 1, 2, \ldots$, we get uncountably many $\theta$'s. Moreover if we put $\theta = \sum_{n=1}^{\infty} \frac{1}{a_n} + r$ so that there exists a positive integer $N$ satisfying $rN \in \mathbb{Z}$, then $\theta$ also satisfies (21) for $n \geq N$ and hence $\theta$ satisfies (8).

The remaining claim in Theorem 1 ii) is clear from (18).

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