Drawing Curves

Toshio Oshima

Abstract We propose a method to determine piecewise cubic Bézier curves passing through given points. Our main purpose is to draw accurate graphs of mathematical functions with smaller data. A program drawing such graphs using our method is realized in a computer algebra and outputs the graphs in a source file of TEX and then transforms it into a PDF file. Our method is also useful for numerical calculation of a given area enclosed by a curve and for numerical integration of functions.

Keywords Bézier curve, cubic spline, computer algebra, Risa/Asir, TEX, Ti*k*Z, 3D graph, numerical integration

1 Introduction

Since the last year I have a class of calculus in my university and show graphs of functions such as $f(x,y) = x^2 - y^2$. I have been developing a library os_muldif.rr [3] of a computer algebra Risa/Asir [5] to realize my research explained in [4] and then I added some functions in the library for such educational purpose including calculus, linear algebra and elementary number theory. The library is an open source and can be equally executed by a personal computer with any one of the operating systems Windows, Mac and UNIX.

In fact, a function in the library executes the procedure in Fig. 1 to get the graphs. Since the PDF file supports cubic Bézier curves, the size of the PDF file obtained in the procedure is usually small and it is independent of the final resolution of the graph.

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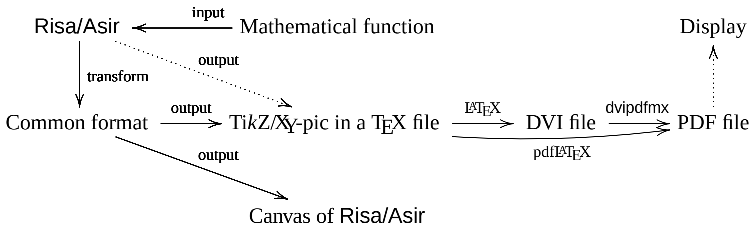
Fig. 1 Procedure



2 Curves

Consider a curve

$$C : [a,b] \ni t \mapsto \gamma(t) = \big(x(t), y(t)\big) \in \mathbb{R}^2. \tag{1}$$

We choose points in $[a,b]$, namely, $P_j = \gamma(t_j) \in C$ with $a = t_0 < t_1 < t_2 < \cdots < t_N = b$ and draw a certain curve C' starting from P_0, exactly passing through $P_1, \ldots, P_{N-1}$ in this order and ending at P_N. We request the following conditions.

- C' is determined only by $\{P_0, P_1, \ldots, P_N\}$.
- C' is a good approximation of C and it is free from its final resolution in drawing.
- Smaller size of data (i.e. the number N) and an output in a popular format are desirable.
- The curve can be described in a usual TEX source file.

One of the way to realize it is to connect the points by cubic Bézier curves and use Ti*k*Z and/or XY-pic which are in a package of a TEX system (cf. Fig. 1).

2.1 Smooth curves

A Bézier curve of degree n is

$$[0,1] \ni t \mapsto P(t) = P(B_0, \ldots, B_n; t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} B_i \tag{2}$$

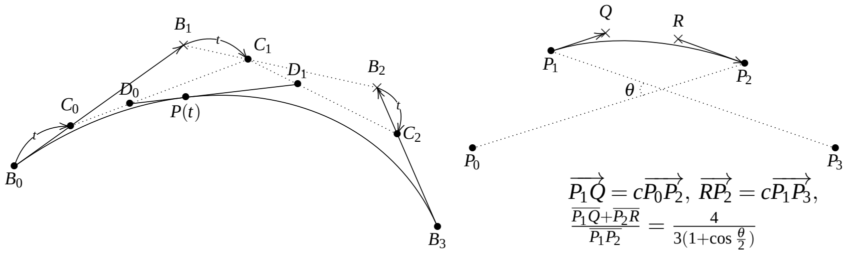
determined by $(n+1)$ points $B_0, \ldots, B_n$.

Note that $P(B, B'; t)$ is the point internally dividing the line segment BB' by $t : 1-t$. Since $P(B_0, \ldots, B_n; t) = P(P(B_0, B_1; t), P(B_1, B_2; t), \ldots, P(B_{n-1}, B_n; t); t)$, the point $P(t)$ is geometrically described. For example, the **cubic Bézier curve** is

$$\begin{aligned} P(t) &= P(B_0, B_1, B_2, B_3; t) = P\big(P(B_0, B_1; t), P(B_1, B_2; t), P(B_2, B_3; t); t\big) \\ &= P\Big(P\big(P(B_0, B_1; t), P(B_1, B_2; t); t\big), P\big(P(B_1, B_2; t), P(B_2, B_3; t); t\big); t\Big). \end{aligned}$$

The curve starts from B_0 to the direction $\overrightarrow{B_0B_1}$ and ends at B_3 to the direction $\overrightarrow{B_2B_3}$. It does not necessarily pass through B_1 nor B_2.

Fig. 2 cubic Bézier curves



Consider a curve C passing through P_0, P_1, P_2, P_3 in this order. We simulate the curve segment of C connecting P_1 to P_2 by the cubic Bézier curve $P(P_1, Q, R, P_2; t)$ with the control points Q and R defined in Fig 2. The number c is determined by

$$c = \frac{4\overline{P_1P_2}}{3(\overline{P_0P_2} + \overline{P_1P_3})} \frac{1}{1 + \sqrt{\frac{1}{2}\left(1 + \frac{(\overrightarrow{P_0P_2}, \overrightarrow{P_1P_3})}{\overline{P_0P_2} \cdot \overline{P_1P_3}}\right)}}. \tag{3}$$

To explain (3) we assume that $\overline{P_0P_1} = \overline{P_1P_2} = \overline{P_2P_3}$ and moreover that $P_0, \ldots, P_3$ are on a circle with the center O. We define a Bézier curve with the control points Q and R which approximates the arc connecting P_1 and P_2. Putting $\angle P_1OP_2 = \theta$, $\overline{OP_1} = r$, $\overline{P_1Q} = \overline{P_2R} = a$, the point T on the Bézier curve corresponding to $t = \frac{1}{2}$ is given as follows under a suitable coordinate system.

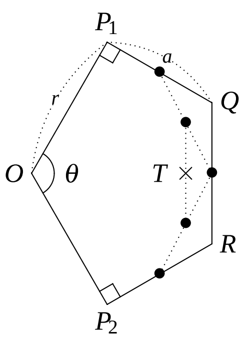
$O : (0, 0),\ P_1 : (r\cos\frac{\theta}{2}, r\sin\frac{\theta}{2}),\ P_2 : (r\cos\frac{\theta}{2}, -r\sin\frac{\theta}{2})$

$Q : (r\cos\frac{\theta}{2} + a\cos\frac{\theta-\pi}{2}, r\sin\frac{\theta}{2} + a\sin\frac{\theta-\pi}{2})$

$\quad = (r\cos\frac{\theta}{2} + a\sin\frac{\theta}{2}, r\sin\frac{\theta}{2} - a\cos\frac{\theta}{2})$

$R : (r\cos\frac{\theta}{2} + a\sin\frac{\theta}{2}, -r\sin\frac{\theta}{2} + a\cos\frac{\theta}{2})$

$T : (r\cos\frac{\theta}{2} + \frac{3}{4}a\sin\frac{\theta}{2}, 0)$



Put $\overline{OT} = \overline{OP_1}$ to approximate the arc. Then

$$r\cos\frac{\theta}{2} + \frac{3}{4}a\sin\frac{\theta}{2} = r$$

and therefore

$$a=\frac{4}{3}\frac{1-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}r=\frac{4}{3}\frac{\sin\frac{\theta}{2}}{1+\cos\frac{\theta}{2}}r.$$

In this case we have

$$\begin{aligned}Q&:\Big(r\cos\tfrac{\theta}{2}+\frac{4}{3}\frac{1-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\sin\tfrac{\theta}{2}r,\ r\sin\tfrac{\theta}{2}-\frac{4}{3}\frac{\sin\frac{\theta}{2}}{1+\cos\frac{\theta}{2}}\cos\tfrac{\theta}{2}r\Big)\\&=\Big((\tfrac{4}{3}-\tfrac{1}{3}\cos\tfrac{\theta}{2})r,\ (1-\tfrac{1}{3}\cos\tfrac{\theta}{2})\frac{\sin\frac{\theta}{2}}{1+\cos\frac{\theta}{2}}r\Big),\\\frac{\overline{P_1Q}}{\overline{P_1P_2}}&=\frac{4}{3}\frac{\sin\frac{\theta}{2}}{1+\cos\frac{\theta}{2}}\frac{1}{2\sin\frac{\theta}{2}}=\frac{2}{3(1+\cos\frac{\theta}{2})}.\end{aligned}\tag{4}$$

Put $r=1$ and $c=\cos\frac{\theta}{2}$. We examine the distance between O and the point

$$B(t)=\big(x(t),y(t)\big)=P_1(1-t)^3+3Qt(1-t)^2+3Rt^2(1-t)+P_2t^3$$

on the Bézier curve. Denoting $t=s+\frac{1}{2}$, we have

$$\begin{aligned}L(s)&:=x(s+\tfrac{1}{2})^2+y(s+\tfrac{1}{2})^2\\&=\frac{16(1-c)^3}{1+c}s^6-\frac{8(1-c)^3}{1+c}s^4+\frac{(1-c)^3}{1+c}s^2+1\\&=\tfrac{(1-c)^3}{1+c}s^2(4s^2-1)^2+1\end{aligned}$$

and when $0\le s\le\frac{1}{2}$,

$$\sqrt[3]{8s^2(1-4s^2)^2}\ge\frac{8s^2+(1-4s^2)+(1-4s^2)}{3}=\frac{2}{3}.$$

The equality in the above holds if and only if $8s^2=1-4s^2$, namely, $s^2=\frac{1}{12}$. Hence $L(s)$ with $|s|\le\frac{1}{2}$ takes the minimal value 1 when $s=0,\pm\frac{1}{2}$ and the maximal value when $s=\pm\frac{1}{2\sqrt{3}}$.

$$\begin{aligned}&L(\pm\tfrac{1}{2\sqrt{3}})-1=\frac{1}{27}\frac{(1-c)^3}{1+c},\\&\sqrt{L(\pm\tfrac{1}{2\sqrt{3}})}-1\fallingdotseq\frac{1}{54}\frac{(1-\cos\frac{\theta}{2})^3}{1+\cos\frac{\theta}{2}}\fallingdotseq\begin{cases}\frac{1}{648} & (\theta=\frac{2\pi}{3}=120^\circ),\\\frac{1}{3668} & (\theta=\frac{\pi}{2}=90^\circ),\\\frac{1}{41900} & (\theta=\frac{\pi}{3}=60^\circ),\\\frac{1}{235541} & (\theta=\frac{\pi}{4}=45^\circ),\\\frac{1}{2683400} & (\theta=\frac{\pi}{6}=30^\circ).\end{cases}\end{aligned}\tag{5}$$

In view of (4), we determine that the segment between P_1 and P_2 in the curve interpolating general P_0,P_1,P_2,P_3 is the cubic Bézier curve with the control points Q and R so that

$$\overrightarrow{P_1Q} = c\overrightarrow{P_0P_2}, \quad \overrightarrow{P_2R} = c\overrightarrow{P_3P_1}, \tag{6}$$

$$\frac{\overline{P_1Q} + \overline{P_2R}}{\overline{P_1P_2}} = \frac{4}{3(1+\cos\frac{\theta}{2})}. \tag{7}$$

Thus

$$\cos\theta = \frac{(\overrightarrow{P_0P_2}, \overrightarrow{P_1P_3})}{\overline{P_0P_2}\cdot\overline{P_1P_3}}, \quad \cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{1}{2}\left(1+\frac{(\overrightarrow{P_0P_2}, \overrightarrow{P_1P_3})}{\overline{P_0P_2}\cdot\overline{P_1P_3}}\right)}$$

$$\text{and} \quad \frac{c\overline{P_0P_2} + c\overline{P_1P_3}}{\overline{P_1P_2}} = \frac{4}{3(1+\cos\frac{\theta}{2})}$$

and therefore we have (3).

The cubic Bézier curve is given by

$$\begin{aligned} B(t) &= P_1(1-t)^3 + 3Qt(1-t)^2 + 3Rt^2(1-t) + P_2t^3 \\ &= (-P_1+3Q-3R+P_2)t^3 + (3P_1-6Q+3R)t^2 + (-3P_1+3Q)t + P_1. \end{aligned}$$

The Catmull-Rom spline curve is defined by

$$\begin{aligned} C(t) = &(-\tfrac{1}{2}P_0 + \tfrac{3}{2}P_1 - \tfrac{3}{2}P_2 + \tfrac{1}{2}P_3)t^3 + (P_0 - \tfrac{5}{2}P_1 + 2P_2 + \tfrac{1}{2}P_3)t^2 \\ &+ (-\tfrac{1}{2}P_0 + \tfrac{1}{2}P_2)t + P_1 \end{aligned}$$

and therefore the corresponding control points Q and R in this case are defined by

$$\begin{cases} Q = P_1 + \frac{1}{6}(P_2 - P_0), \\ R = P_2 + \frac{1}{6}(P_3 - P_1), \end{cases}$$

which means that we fix $c = \frac{1}{6}$ in (6).

In our case, the relative error $\left|\frac{\overline{OB(t)}}{\overline{OP_1}} - 1\right|$ is less than $\frac{1}{640}$ (resp. $\frac{1}{3600}$) if $\angle P_1OP_2 \leq 120^\circ$ (resp. $\leq 90^\circ$). Note that a Bézier curve never coincides with an exact arc.

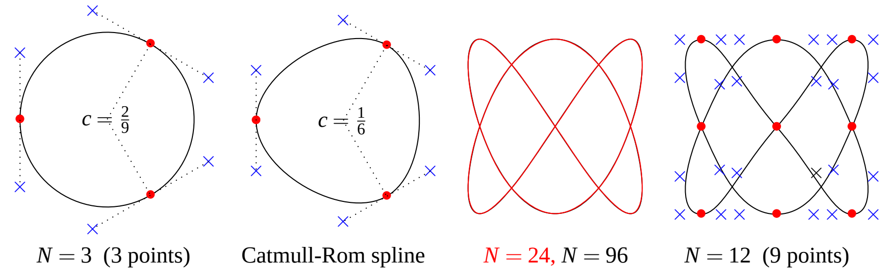
For a closed curve C passing through points $R_0, R_1, \ldots, R_N = R_0$ in this order we draw a curve segment between R_j and R_{j+1} by putting $P_i = R_{i+j-1}$ for $i = 0, 1, 2$ and 3 as in the above and $R_{\nu\pm N} = R_\nu$ $(\nu = 1, \ldots, N)$. Then the resulting curve C' we draw is a smooth closed curve (of class C^1) which simulates C.

When the number c is fixed to be $\frac{1}{6}$ in (6), the corresponding curve is known as the (uniform) Catmull-Rom spline curve (cf. [2]). It is invariant under affine transformations and our curve is invariant under conformal affine transformations.

The following first example in Fig. 3 is the curve drawn by the three points $(\cos t, \sin t)$ with $t = \pm\frac{\pi}{3}, \pi$ indicated by •. The other 6 points calculated by using (3) are indicated by ×. In the final PDF file the positions of these 9 points are only written and the real rendering of the Bézier curve is done by a viewer of the file and therefore the size of the PDF file is small. The second example is the (uniform/centripetal) Catmull-Rom spline curve passing through these three points.

The other examples in Fig. 3 are the Lissajous curve $\gamma(t) = (\sin 2t, \sin 3t)$ drawn by the points corresponding to $t = \frac{2\pi j}{N}$ for $j = 0, \dots, N$.

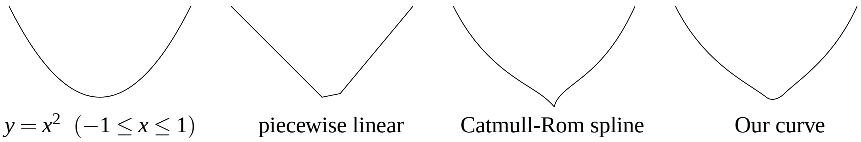
Fig. 3 Bézier curves



If the points $P_j = \gamma(t_j)$ are not suitably chosen, the resulting curve drawn by the points may be not good. Even in this case our curve is better than the corresponding Catmull-Rom spline curve as in the following example.

Suppose we draw a graph of the parabola defined by $y = x^2$. Taking the points on the curve $\gamma(t) = (t, t^2)$ corresponding to $t = -2, -1, 0, 0.2, 1, 2$, we draw curve for $-1 \le t \le 1$ by these 6 points.

Fig. 4 Parabola



To avoid a singularity or a loop in a Bézier segment, a generalization of Catmul-Rom spline is introduced (cf. [1]):

$$
\begin{aligned}
\gamma(t) &= \frac{t_2 - t}{t_2 - t_1} B_1 + \frac{t - t_1}{t_2 - t_1} B_2 \qquad (t \in [t_1, t_2]),\\
B_1 &= \frac{t_2 - t}{t_2 - t_0} B_1 + \frac{t - t_0}{t_2 - t_0} B_2, \quad B_1 = \frac{t_3 - t}{t_3 - t_1} B_1 + \frac{t - t_1}{t_3 - t_1} B_2,\\
A_1 &= \frac{t_1 - t}{t_1 - t_0} P_0 + \frac{t - t_0}{t_1 - t_0} P_1, \quad A_2 = \frac{t_2 - t}{t_2 - t_1} P_1 + \frac{t - t_1}{t_2 - t_1} P_2,\\
A_3 &= \frac{t_3 - t}{t_3 - t_2} P_2 + \frac{t - t_2}{t_3 - t_2} P_3, \quad t_j = (\overline{P_{j-1} P_j})^{\alpha} + t_{j-1} \qquad (j = 1, 2, 3).
\end{aligned}
$$

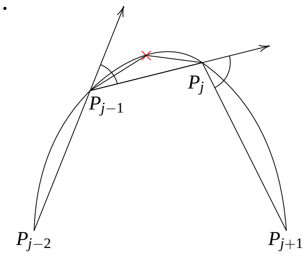
If $\alpha = 0$, the above curve equals the standard (uniform) Catmul-Rom spline. When $\alpha = 1$, the curve is called chordal Catmul-Rom spline. When $\alpha = 0.5$, the curve is called centripetal Catmull-Rom spline and has more desirable properties compared to the original one (cf. [6]). It will not form loop nor cusp within a curve segment.

But these Catmul-Rom splines produce the same result as in Fig. 4 for the points equally distributed on a circle because $\overline{P_{j-1}P_j}$ does not depend on j.

2.2 Singularities

We consider a curve $\gamma(t)$ $(t \in [a,b])$ which has singular points or discontinuous points. We assume that the curve is a finite union of smooth curves but we do not know the singular points of the curve.

First we choose points $P_j = \gamma(t_j)$ with $t_0 = a < t_1 < \cdots < t_N = b$ on the curve. We put $t_j = a + \frac{j(b-a)}{N}$ in most cases (or as default)[1].



For every j, add the point $\gamma(\frac{t_{j-1}+t_j}{2})$ if

- $\frac{(\overrightarrow{P_{j-2}P_{j-1}}, \overrightarrow{P_{j-1}P_j})}{\overline{P_{j-2}P_{j-1}} \cdot \overline{P_{j-1}P_j}} < C_1$ or $\frac{(\overrightarrow{P_{j-1}P_j}, \overrightarrow{P_jP_{j+1}})}{\overline{P_{j-1}P_j} \cdot \overline{P_jP_{j+1}}} < C_1$ or
- $\overline{P_{j-1}P_j} > C_2$

Repeat the above up to m times,

If the length $\overline{P_{j-1}P_j}$ still exceeds a given threshold value C_2 after this procedure, we cut our curve between two points P_{j-1} and P_j.

The default threshold values are $C_1 = \cos 30^\circ$, $C_2 = \frac{\text{diameter of Window}}{16}$ and $m = 4$.

We examine the graph of the function

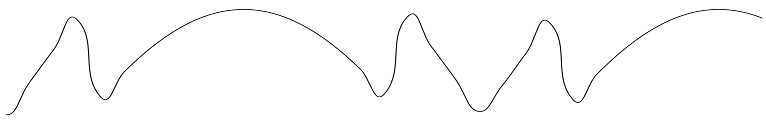
$$y = |2\sin x| - \left[|2\sin x|\right] \quad (0 \le x \le 5).$$

Here for a real number t, $\left[t\right]$ denotes the largest integer which does not exceed t.

Note that this function is discontinuous at $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$ and not smooth at $x = \pi$.

If we do not care the singularities, we have Fig. 5.

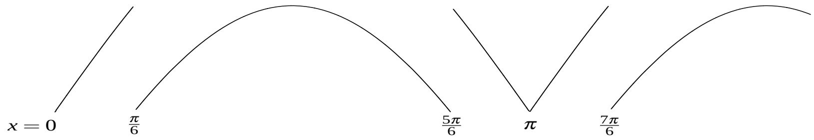
Fig. 5 $y = \left||2\sin x| - \left[|2\sin x|\right]\right|$ $(0 \le x \le 5$, $m = 0$ and $N = 32)$



The procedure explained in this subsection gives Fig. 6 and the number of segments of Bésier curves increases from 32 to 70.

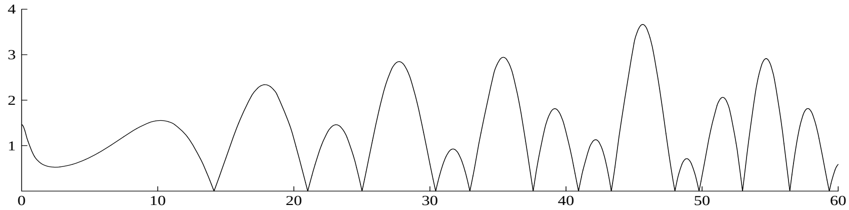
[1] Moreover if the curve is defined outside $[a,b]$, we use the points P_{-1} and P_{N+1} to define Bézier curves.

Fig. 6 $y = \big||2\sin x| - [|2\sin x|]\big|$ $(0 \le x \le 5$, $m = 4$ and $N = 32 \to 70)$



The graph of the absolute value of Riemann's zeta function $\zeta(z)$ for $\mathrm{Re}\, z = \frac{1}{2}$ is given in Fig. 7. Risa/Asir takes less than a second to get it in a PDF file.

Fig. 7 $y = \big|\zeta(\frac{1}{2} + x\sqrt{-1})\big|$ $(m = 6$ and $N = 96 \to 355)$

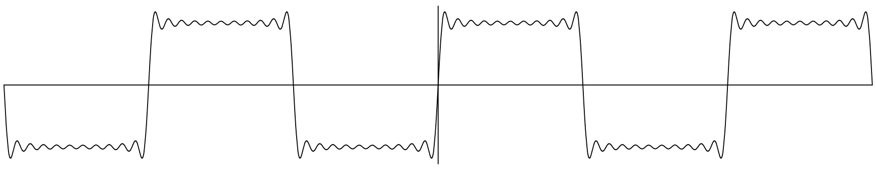


The final example in this subsection is the finite Fourier series

$$y = \sin x + \frac{1}{3}\sin\frac{x}{3} + \frac{1}{5}\sin\frac{x}{5} + \cdots + \frac{1}{21}\sin\frac{x}{21}$$

which approximates a square wave.

Fig. 8 Fourier series $(m = 6$ and $N = 192 \to 1020)$



3 Applications

3.1 Circles, arcs and ovals

The relative error of our approximation of an arc by a cubic Bézier curve becomes smaller when its central angle becomes smaller. If the angle is smaller than 120° (resp. 90°), then it is smaller than 0.16% (resp. 0.028%) as is shown in the previous section. The relative error here is measured by the distance from the center of the circle containing the arc.

Hence the central angle of an arc is not large, it is sufficient for us to approximates it by a single cubic Bézier curve or at most three cubic segments for most purposes.

Moreover since the Bézier curve is compatible with affine transformations, we can also draw an oval and an arc of an oval with the same accuracy by using an affine transformation of our approximation of a circle or an arc of a circle. These are realized in [3].

3.2 Integration

The area enclosed by a curve is numerically calculated by our approximation since an area enclosed by a curve with cubic Bézier segments is easily calculated.

Suppose an area is enclosed by segments of cubic Bézier curves

$$[0,1] \ni t \mapsto \gamma_j(t) = \big(x_j(t), y_j(t)\big) \qquad (j = 0, \ldots, N).$$

Then the absolute value of the line integral

$$I(\gamma) = \sum_{j=0}^{N} \int_0^1 y_j(t)\, dx_j(t) = \sum_{j=0}^{N} \int_0^1 x_j'(t) \cdot y_j(t)\, dt$$

gives the area. Here $x_j'(t) \cdot y_j(t)$ are polynomials of degree 5 and therefore the above value is easily calculated.

If the curve is an approximation of the graph of $y = f(x)$ with $x \in [a,b]$, the above value is an approximation of $\int_a^b f(x)\, dx$.

In the following table we show examples of the relative errors of the numerical integrations using this method. In the table, circle and cardioid are parametrized by

$$(\cos\theta, \sin\theta) \ \text{ and } \ \big((1+\cos\theta)\cos\theta, (1+\cos\theta)\sin\theta\big),$$

respectively. For example, in the case of cardioid in the table, “32 parts” means that the cardioid is approximated by 32 cubic Bézier segments determined only by the points $\big((1+\cos\theta_j)\cos\theta_j, (1+\cos\theta_j)\sin\theta_j\big)$ with $\theta_j = \frac{j\pi}{16} - \pi$ and $j = 0, 1, \ldots, 32$ and the approximated area is calculated by the segments.

Integrations using Bézier curves

curve	interval	16 parts	32 parts	96 parts	384 parts	1536 parts
circle	$0 \le \theta \le 2\pi$	6.8×10^{-8}	1.1×10^{-9}	1.5×10^{-12}	3.2×10^{-17}	8.7×10^{-20}
cardioid	$-\pi \le \theta \le \pi$	5.4×10^{-4}	3.1×10^{-5}	3.8×10^{-7}	1.5×10^{-9}	5.8×10^{-12}
$x\sin x$	$0 \le x \le \pi$	2.9×10^{-4}	1.8×10^{-6}	2.2×10^{-8}	8.7×10^{-11}	3.4×10^{-13}
$\frac{\sin x}{x}$	$0 < x \le \pi$	1.5×10^{-6}	9.5×10^{-8}	1.2×10^{-9}	4.6×10^{-12}	1.7×10^{-14}
$\frac{1}{x^2+1}$	$-\infty < x < \infty$	1.3×10^{-5}	1.3×10^{-7}	8.5×10^{-10}	4.7×10^{-12}	2.1×10^{-14}
e^{-x^2}	$-\infty < x < \infty$	7.1×10^{-4}	1.3×10^{-4}	2.6×10^{-6}	1.1×10^{-8}	4.3×10^{-11}
$x^{-\frac{3}{2}}$	$1 \le x < \infty$	3.0×10^{-4}	3.8×10^{-5}	1.4×10^{-6}	6.6×10^{-9}	2.6×10^{-11}
$\frac{1}{x^2+\sqrt{-1}}$	$-\infty < x < \infty$	2.3×10^{-3}	1.7×10^{-4}	2.6×10^{-6}	1.9×10^{-8}	8.1×10^{-11}
$e^{\frac{1}{z}}$	$\lvert z\rvert = 1$	7.6×10^{-5}	4.1×10^{-6}	4.8×10^{-8}	1.9×10^{-10}	7.3×10^{-13}

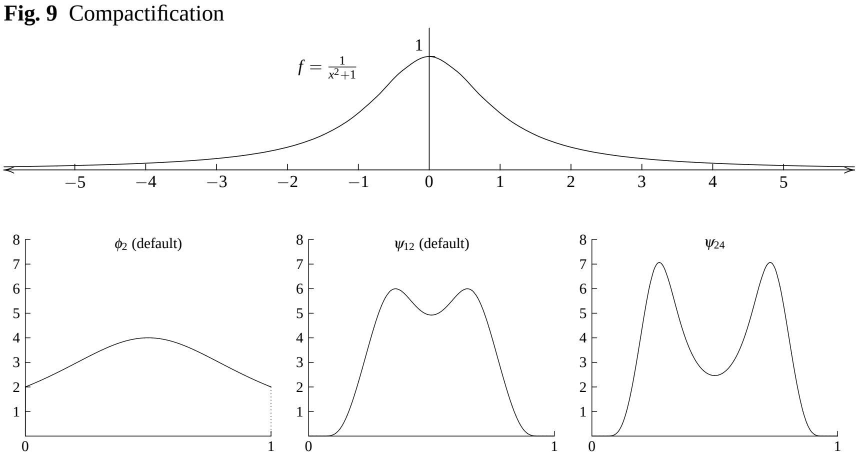
If the interval of integration is infinite, we compactify it to $[0,1]$ for the calculation. For example, if the interval is $(-\infty,\infty)$, the transformations

$$\phi_C : (0,1) \ni t \mapsto x = \tfrac{1}{C}\big(\tfrac{1}{1-t} - \tfrac{1}{t}\big) \in (-\infty,\infty),$$
$$\psi_C : (0,1) \ni t \mapsto x = \tfrac{1}{C}\big(e^{\frac{1}{1-t}} - e^{\frac{1}{t}}\big) \in (-\infty,\infty).$$

are used in [3]. In the above examples, the positive constant C is the default value in [3]. If $|f(x)| = O(x^{-2})$, the transformation by ϕ_C usually gives a better approximation than by ψ_C.

In Fig. 9 we show the change of integrand of $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+1}$ under the compactification.

Fig. 9 Compactification



3.3 3D graphs

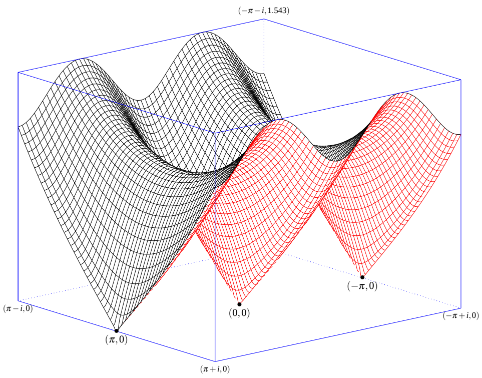
Our original main purpose is to draw graphs of surfaces defined by $z = f(x,y)$ with mathematical functions $f(x,y)$. Using our method Fig. 1, we draw curves on a surface defined by the condition that x is constant or y is constant. It takes $10 \sim 30$ seconds to get a required PDF file after a command in Risa/Asir if $f(x,y)$ is a simple rational function. We can use Ti*k*Z and XY-pic. In contrast to XY-pic the source text in Ti*k*Z is more readable, easy to be edited and has stronger abilities such that it supports coloring and filling region by a pattern but is takes a little longer time to be transformed into a PDF file. Hence our library [3] supports both of them.

We give two examples $z = |\sin(x+y\sqrt{-1})|$ and $z = \frac{xy^2}{x^2+y^4}$:

Fig. 10 3D graphs

$|\sin z| \quad (z = x+yi,\ -\pi \le x \le \pi,\ -1 \le y \le 1)$

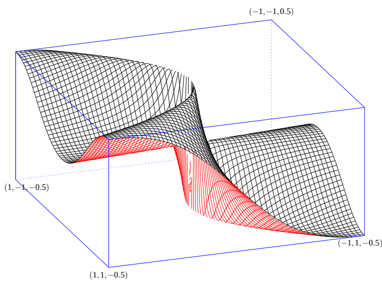
angle $(50^\circ, 15^\circ)$ ratio $1:3:3$



$z = \frac{xy^2}{x^2+y^4} \quad (-1 \le x \le 1, -1 \le y \le 1)$

angle $(70^\circ, 20^\circ)$

This function is discontinuous at $(x,y) = (0,0)$



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