Commuting differential operators with regular singularities

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Summary. We study a system of partial differential equations defined by commuting family of differential operators with regular singularities. We construct ideally analytic solutions depending on a holomorphic parameter. We give some explicit examples of differential operators related to $SL(n, \mathbb{R})$ and completely integrable quantum systems.

Key words: differential operators, regular singularity, completely integrable systems

Dedicated to Professor Takahiro Kawai on the occasion of his 60th birthday.

1 Introduction

The invariant differential operators on a semisimple symmetric space have regular singularities along the boundaries of the space which is realized in a compact manifold by [O6]. In the case of a Riemannian symmetric space G/K, the study of such operators in [KO] enables [K–] to have the Poisson integral expression of any simultaneous eigenfunction of the operators. Here G is a connected real semisimple Lie group with finite center and K is its maximal compact subgroup.

In the case of the group manifold G, which is an example of a semisimple symmetric space, Harish-Chandra gives an asymptotic expansion of a right and left K-finite eigenfunction, which plays an important role in the harmonic analysis on G (cf. [Ha]). He uses only the Casimir operator to get the asymptotic expansion, which suggests us that one operator controls other operators together with some geometry.

On the other hand, the Schrödinger operator corresponding to Calogero-Moser-Sutherland system with a trigonometric potential function (cf. [Su]) or a Toda finite chain (cf. [To]) is completely integrable and the integrals with higher orders are uniquely characterized by the Schrödinger operator and so are the simultaneous eigenfunctions. These integrals also have regular singularities at infinity.

In this note we study a general commuting system of differential operators with regular singularities by paying attention to the fact that an operator characterizes the system. Our argument used in this note is based on expansions in power series and hence it is rather elementary compared to that in [KO] and [O4] where a microlocal method is used.

In fact we will study matrices of differential operators which may not commute with others in the system but satisfy a certain condition because it is better to do so even in the study of commuting scalar differential operators. Some of its reasons will be revealed in the proof of Theorem 4.1, that of Theorem 6.3, Remark 4.3 ii) etc.

In §2 we study differential operators which commute one operator. We will see that the *symbol map* σ_* plays an important role. In the case of the first example above the map corresponds to Harish-Chandra's isomorphism of the invariant differential operators. In the case of the Schrödinger operator above it corresponds to the commutativity among the integrals with higher orders.

In §3 we construct some of multivalued holomorphic solutions of the system around the singular points which we call *ideally analytic solutions* and then in §4 we study the *induced equations* of other operators, which assures that the solutions automatically satisfy some other differential equations.

In §5 we study the holonomic system of differential equations with constant coefficients holomorphically depending on a parameter, which controls the *leading terms* of the ideally analytic solutions.

In §6 we study a *complete system of differential equations with regular singularities* which means that the system is sufficient to formulate a boundary value problem along the singularities and we describe all the ideally analytic solutions. In particular, when the system has a holomorphic parameter, we construct solutions depending holomorphically on the parameter. It is in fact useful to introduce a parameter for the study of a specific system by holomorphically deforming it to generic simpler ones.

In §7 and §8 we give some explicit examples of the systems related to $SL(n,\mathbb{R})$ and the completely integrable quantum systems with regular singularities at infinity, respectively. Moreover we give Theorem 8.1 in the case of completely integrable quantum systems with two variables.

2 Commuting differential operators with regular singularities

For a positive integer m and a ring R we will denote by $M(m,R)$ the ring of square matrices of size m with components in R and by $R[\xi]$ the ring of polynomials of n indeterminates $\{\xi_1,\dots,\xi_n\}$ if $\xi=(\xi_1,\dots,\xi_n)$. The (i,j)-component of $A\in M(m,R)$ is denoted by A_{ij} and we naturally identify $M(1,R)$ with R.

Let M be an $(n+n')$-dimensional real analytic manifold and let N_i be one-codimensional submanifolds of M such that $N_1, \ldots, N_n$ are normally crossing at $N = N_1 \cap \cdots \cap N_n$. We assume that M and N are connected. We will fix a local coordinate system $(t, x) = (t_1, \ldots, t_n, x_1, \ldots, x_{n'})$ around a point $x^o \in N$ so that N_i are defined by the equations $t_i = 0$, respectively.

Let $\mathcal{A}_N$ denote the space of real analytic functions on N and $\mathcal{A}_M$ the space of real analytic functions defined on a neighborhood of N in M. For $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n$ we put

$$|\alpha| = \alpha_1 + \cdots + \alpha_n,$$
$$\alpha < \beta \;\Leftrightarrow\; \alpha_i \leq \beta_i \quad \text{for } i = 1, \ldots, n \text{ and } \alpha \neq \beta.$$

Let $\mathbb{N}$ be the set of non-negative integers. We will denote

$$\begin{cases} \vartheta_i = t_i \frac{\partial}{\partial t_i}, \quad \partial_x = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n'}}\right), \\ \vartheta^\alpha = \vartheta_1^{\alpha_1} \cdots \vartheta_n^{\alpha_n} \quad \text{for } \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n, \\ \partial_x^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_{n'}^{\beta_{n'}}} \quad \text{for } \beta = (\beta_1, \ldots, \beta_{n'}) \in \mathbb{N}^{n'}, \\ t^\lambda = t_1^{\lambda_1} \cdots t_n^{\lambda_n} \quad \text{for } \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n. \end{cases}$$

Let $\mathcal{D}_M$ and $\mathcal{D}_N$ denote the rings of differential operators on M and N with coefficients in $\mathcal{A}_M$ and $\mathcal{A}_N$, respectively.

Definition 2.1. *Let $\widetilde{\mathcal{D}}_*$ denote the subring of $\mathcal{D}_M$ whose elements P have the form*

$$P = \sum_{(\alpha,\beta)\in\mathbb{N}^{n+n'}} a_{\alpha,\beta}(t,x)\vartheta^\alpha \partial_x^\beta \quad \text{with} \quad a_{\alpha,\beta}(t,x) \in \mathcal{A}_M. \tag{1}$$

Here the sum above is finite. Moreover $\mathcal{D}_*$ denotes the subring of $\widetilde{\mathcal{D}}_*$ whose elements P of the form (1) *satisfy*

$$a_{\alpha,\beta}(0,x) = 0 \quad \text{if} \quad \beta \neq 0. \tag{2}$$

When P is an element of $\mathcal{D}_*$, P is said to have regular singularities in the weak sense along the set of walls $\{N_1, \ldots, N_n\}$ with the edge N (cf. [KO]).

Let define a map σ_* of $\widetilde{\mathcal{D}}_*$ to $\mathcal{D}_N[\xi]$ by

$$\sigma_*(P)(x,\xi,\partial_x) := \sum_{\alpha\in\mathbb{N}^n,\,\beta\in\mathbb{N}^{n'}} a_{\alpha,\beta}(0,x)\xi^\alpha \partial_x^\beta$$

for P in (1)*. Then*

$$t^{-\lambda} P t^\lambda \phi(t,x)\,\Big|_{t=0} = \sigma_*(P)(x,\lambda,\partial_x)\phi(0,x) \quad \text{for } \phi \in \mathcal{A}_M \text{ and } \lambda \in \mathbb{C}^n.$$

Here we note that the condition $P \in \widetilde{\mathcal{D}}_*$ equals

$$t^{-\lambda} P t^\lambda \phi(t,x) \in \mathcal{A}_M \quad \text{for} \quad \forall\phi(t,x) \in \mathcal{A}_M$$

and σ_* is a ring homomorphism of $\widetilde{\mathcal{D}}_*$ to $\mathcal{D}_N[\xi]$ and $\sigma_*(\mathcal{D}_*) = \mathcal{A}_N[\xi]$.

For $k \in \mathbb{N}$ and $P \in \widetilde{\mathcal{D}}_*$ with the form (1) we put

$$\sigma_k(P)(t,x,\xi,\tau) := \sum_{|\alpha|+|\beta|=k} a_{\alpha,\beta}(t,x)\xi^\alpha\tau^\beta$$

and then the order of P, which is denoted by $\operatorname{ord} P$, is the maximal integer k with $\sigma_k(P) \neq 0$.

For $P = \Bigl(P_{ij}\Bigr)_{\substack{1\le i\le m\\ 1\le j\le m}} \in M(m, \widetilde{\mathcal{D}}_*)$, the order of P is defined to be the maximal order of the components of P and denoted by $\operatorname{ord} P$. We put

$$\begin{aligned}
\sigma(P) &:= \Bigl(\sigma_{\operatorname{ord} P}(P_{ij})\Bigr)_{\substack{1\le i\le m\\ 1\le j\le m}} \in M(m, \mathcal{A}_M[\xi,\tau]),\\
\sigma_*(P) &:= \Bigl(\sigma_*(P_{ij})\Bigr)_{\substack{1\le i\le m\\ 1\le j\le m}} \in M(m, \mathcal{D}_N[\xi]),\\
\bar\sigma_*(P) &:= \sigma(P)(0,x,\xi,\partial_x) \in M(m, \mathcal{D}_N[\xi]).
\end{aligned}$$

Then as a polynomial of ξ, $\bar\sigma_*(P)$ is the homogeneous part of $\sigma_*(P)$ whose degree equals $\operatorname{ord} P$. For P, $Q \in \widetilde{\mathcal{D}}_*$, we note that $\sigma(PQ) = \sigma(P)\sigma(Q)$ and

$$\begin{aligned}
\sigma_{\operatorname{ord} P+\operatorname{ord} Q-1}([P,Q]) = \sum_{i=1}^{n}\Bigl(\frac{\partial\sigma(P)}{\partial\xi_i}t_i\frac{\partial\sigma(Q)}{\partial t_i} - \frac{\partial\sigma(Q)}{\partial\xi_i}t_i\frac{\partial\sigma(P)}{\partial t_i}\Bigr)\\
+\sum_{j=1}^{n'}\Bigl(\frac{\partial\sigma(P)}{\partial\tau_j}\frac{\partial\sigma(Q)}{\partial x_j} - \frac{\partial\sigma(Q)}{\partial\tau_j}\frac{\partial\sigma(P)}{\partial x_j}\Bigr).
\end{aligned}$$

Theorem 2.2. *Let P and Q be nonzero elements of $M(m, \widetilde{\mathcal{D}}_*)$ such that $[P,Q]=0$, $P \in M(m,\mathcal{D}_*)$ and $\sigma(P)$ is a scalar matrix satisfying*

$$\sum_{\nu=1}^{n}\gamma_\nu\frac{\partial\bar\sigma_*(P)}{\partial\xi_\nu} \not\equiv 0 \quad \text{for} \quad \forall\gamma \in \mathbb{N}^n \setminus \{0\}. \tag{3}$$

Here "$\not\equiv 0$" means "not identically zero". Suppose that $\sigma_{\operatorname{ord} P-1}(P)$ or $\sigma(Q)$ is a scalar matrix. Then $[\sigma_*(P), \sigma_*(Q)] = 0$ and $\bar\sigma_*(Q) \neq 0$. Moreover if $\sigma(P)(t,x,\xi,\tau)$ does not depend on t, so does $\sigma(Q)(t,x,\xi,\tau)$.

Proof. Since σ_* is an algebra homomorphism, $[\sigma_*(P),\sigma_*(Q)] = \sigma_*([P,Q]) = 0$.

Put $r_P = \operatorname{ord} P$ and $r_Q = \operatorname{ord} Q$. Fix i and j such that $\sigma_{r_Q}(Q_{ij}) \neq 0$. Note that the assumption implies

$$\sigma_{r_P+r_Q-1}([P,Q]_{ij}) = \sigma_{r_P+r_Q-1}([P_{11},Q_{ij}]).$$

Put

$$\sigma_{r_P}(P_{11}) = \sum_{\substack{\beta,\gamma\\ |\beta|\le r_P}} p_{\beta,\gamma}(x,\xi)t^\gamma\tau^\beta,\ \ \sigma_{r_Q}(Q_{ij}) = \sum_{\substack{\beta,\gamma\\ |\beta|\le r_Q}} q_{\beta,\gamma}(x,\xi)t^\gamma\tau^\beta,$$

$$\sigma_{r_P+r_Q-1}([P_{11},Q_{ij}]) = \sum_{\substack{\beta,\gamma \\ |\beta|\le r_P+r_Q-1}} s_{\beta,\gamma}(x,\xi)t^\gamma\tau^\beta$$

and choose $(\beta^o,\gamma^o)\in\mathbb{N}^{n'+n}$ such that

$$\begin{cases} q_{\beta^o,\gamma^o}\neq 0, & \\ q_{\beta,\gamma}=0 & \text{if } \gamma<\gamma^o,\\ q_{\beta,\gamma^o}=0 & \text{if } \beta>\beta^o.\end{cases}$$

Then

$$s_{\beta^o,\gamma^o}t^{\gamma^o}\tau^{\beta^o} = \Bigl(\sum_{\nu=1}^n \frac{\partial p_{0,0}}{\partial\xi_\nu}\gamma^o_\nu\Bigr)\Bigl(q_{\beta^o,\gamma^o}t^{\gamma^o}\tau^{\beta^o}\Bigr), \tag{4}$$

which proves the first claim in the theorem because the condition $[P,Q]=0$ with the assumption of the theorem means $\gamma^o=0$.

Moreover suppose $p_{\beta,\gamma}=0$ for $\gamma\neq 0$. Then (4) is valid for any $\gamma^o\in\mathbb{N}^n$ and $\beta^o\in\mathbb{N}^{n'}$ satisfying $q_{\beta,\gamma^o}=0$ for $\beta>\beta^o$ and hence the condition $[P,Q]=0$ means $q_{\beta^o,\gamma^o}=0$ if $\gamma^o\neq 0$. Thus $q_{\beta,\gamma^o}=0$ if $\gamma^o\neq 0$. □

Corollary 2.3. *Let $P\in M(m,\mathcal{D}_*)$ such that $\sigma(P)$ and $\sigma_{\operatorname{ord}P-1}(P)$ are scalar matrices. Suppose $\bar\sigma_*(P)$ satisfies* (3)*. Then the map*

$$\sigma_*:\ M(m,\widetilde{\mathcal{D}}_*)^P := \{Q\in M(m,\widetilde{\mathcal{D}}_*)\,;\,[P,Q]=0\}\to M(m,\mathcal{D}_N[\xi]),\qquad Q\mapsto\sigma_*(Q)$$

is an injective algebra homomorphism.

In particular, when $m=1$, $\mathcal{D}_*^P:=\{Q\in\mathcal{D}_*\,;\,[P,Q]=0\}$ is commutative.

Proof. Since σ_* is an algebra homomorphism and the condition $Q_1,Q_2\in M(m,\widetilde{\mathcal{D}}_*)^P$ implies $[Q_1,Q_2]\in M(m,\widetilde{\mathcal{D}}_*)^P$, this corollary is a direct consequence of Theorem 2.2. □

Remark 2.4. i) Retain the notation in Theorem 2.2. Then (3) is valid for $P\in M(m,\mathcal{D}_*)$ if n functions $\frac{\partial\bar\sigma_*(P)}{\partial\xi_1},\ldots,\frac{\partial\bar\sigma_*(P)}{\partial\xi_n}$ are linearly independent over $\mathbb{R}$. In particular, if $\operatorname{ord}P=2$ and $\bar\sigma_*(P)$ is a scalar matrix, the condition that

$$\text{the matrix } \left(\frac{\partial^2\bar p}{\partial\xi_i\partial\xi_j}\right)_{\substack{1\le i\le n\\ 1\le j\le n}} \text{ is invertible for generic } x\in N$$

implies (3). Here $\bar p$ is the diagonal element of $\bar\sigma_*(P)$.

ii) The assumption $P\in M(m,\mathcal{D}_*)$ is necessary in Theorem 2.2. For example, $[t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x},t\frac{\partial}{\partial x}]=0$ and $\sigma_*(t\frac{\partial}{\partial x})=0$. Moreover we note that

$$\left[\begin{pmatrix} t\frac{\partial}{\partial t} & 0\\ 0 & t\frac{\partial}{\partial t}+1\end{pmatrix},\begin{pmatrix}0 & t\\ 0 & 0\end{pmatrix}\right]=0.$$

This gives an example such that $\sigma_{\mathrm{ord}\,P-1}(P)$ and $\sigma(Q)$ are not scalar matrices.

iii) The invariant differential operators on a Riemannian symmetric space G/K of non-compact type have regular singularities along the boundaries of a realization of the space constructed by [O2] and the map σ_* of $\mathcal{D}_*^P$ to $\mathcal{A}_N[\xi]$ in Corollary 2 corresponds to Harish-Chandra isomorphism (cf. [K–]).

The element of the universal enveloping algebra $U(\mathfrak{g})$ of the Lie algebra of G defines a differential operator on the realization of G/K through the infinitesimal action of the left translation by elements of G. Then the differential operator is an element of $\widetilde{\mathcal{D}}_*$.

Moreover the invariant differential operators on a semisimple symmetric space whose rank is larger than its real rank are in $\widetilde{\mathcal{D}}_*$ (cf. [O4]).

The radial parts of the Casimir operator acting on K-finite sections of certain homogeneous vector bundle of G satisfy the assumption of Theorem 2.2 (cf. (26) and (27) for examples).

3 Ideally analytic solutions without logarithmic terms

For a subset Σ of $\mathbb{N}^n$ define

$$\overline{\Sigma} := \bigl\{\alpha \in \mathbb{N}^n\,;\, \{\alpha+\gamma\,;\,\gamma\in\mathbb{N}^n\} \cap \Sigma \neq \emptyset\bigr\},$$

$$\partial\Sigma := \Bigl\{\alpha \in \mathbb{N}^n \setminus \overline{\Sigma}\,;\ \text{there exists } \gamma \in \bar{\Sigma} \text{ such that } \sum_{i=1}^n |\alpha_i - \gamma_i| = 1\Bigr\}.$$

Moreover we denote by $\hat{\mathcal{A}}_M$ the ring of formal power series of $t = (t_1, \dots, t_n)$ with coefficients in $\mathcal{A}_N$.

Theorem 3.1. *Let $P \in M(m, \mathcal{D}_*)$.*

i) *Let Σ be a subset of $\mathbb{N}^n$ such that*

$$\det\bigl(\sigma_*(P)(x,\gamma)\bigr) \not\equiv 0 \quad \textit{for} \ \ \forall\gamma \in \mathbb{N}^n \setminus \Sigma.$$

Let $\hat{u}(t,x) = \sum_{\alpha\in\mathbb{N}^n} u_\alpha(x)t^\alpha \in \hat{\mathcal{A}}_M^m$ be a formal solution of $P\hat{u} = 0$. Then $\hat{u} = 0$ if $u_\alpha = 0$ for $\forall\alpha \in \Sigma$.

Hereafter in this theorem suppose

$$\det \bar{\sigma}_*(P)(x,\xi) \neq 0 \quad \textit{for}\ \forall\xi = (\xi_1, \dots, \xi_n) \in [0,\infty)^n \setminus \{0\} \ \textit{and}\ \forall x \in N. \tag{5}$$

ii) *If $\hat{u} \in \hat{\mathcal{A}}_M^m$ satisfies $P\hat{u} \in \mathcal{A}_M^m$, then $\hat{u} \in \mathcal{A}_M^m$.*

iii) *Fix $f \in \mathcal{A}_M^m$, a point $x^o \in N$ and a finite subset Σ of $\mathbb{N}^n$ such that*

$$\det\bigl(\sigma_*(P)(x^o,\gamma)\bigr) \neq 0 \quad \textit{for} \ \ \forall\gamma \in \mathbb{N}^n \setminus \Sigma.$$

By shrinking $M \ni x^o$ if necessary and denoting

$$\mathcal{A}_M(P^{-1}f) := \{u \in \mathcal{A}_M^m\,;\, Pu = f\},$$

$$\mathcal{A}_M(P^{-1}f)^{\Sigma} := \{\bar{u} = \sum_{\alpha\in\overline{\Sigma}} u_\alpha(x)t^\alpha \in \mathcal{A}_M^m\,;\, P\bar{u} \equiv f \mod \sum_{\beta\in\partial\Sigma} \mathcal{A}_M^m t^\beta\},$$

the natural restriction map

$$\mathcal{A}_M(P^{-1}f) \xrightarrow{\sim} \mathcal{A}_M(P^{-1}f)^{\Sigma}, \quad \sum_{\alpha\in\mathbb{N}^n} u_\alpha(x)t^\alpha \mapsto \sum_{\alpha\in\overline{\Sigma}} u_\alpha(x)t^\alpha$$

is a bijection. Here in particular

$$\mathcal{A}_M(P^{-1}f)^{\{0\}} = \{u \in \mathcal{A}_N^m\,;\, \sigma_*(P)\big(x,0)u = f|_{t=0}\}.$$

Proof. The proof proceeds in a similar way as in [O3, Theorem 2.1] where we studies the same problem with $n = 1$.

We may assume $x^o = 0$. Expanding functions in convergent power series of (t,x) at $(0,0)$, we will prove the theorem in a neighborhood of $(0,0)$.

Put $r = \operatorname{ord} P$ and

$$P = \sigma_*(P)(x,\vartheta) + \sum_{\substack{(\alpha,\beta)\in\mathbb{N}^{n+n'}\\ |\alpha|+|\beta|\le r}} p_{\alpha,\beta}(t,x)\vartheta^\alpha\partial_x^\beta.$$

Then $p_{\alpha,\beta}(0,x) = 0$. For a finite subset $\Sigma \subset \mathbb{N}^n$ and

$$\hat{u}(t,x) = \sum_{\alpha\in\mathbb{N}^n} \hat{u}_\alpha(x)t^\alpha \in \hat{\mathcal{A}}_M^m,$$

put

$$\bar{u}(t,x) = \sum_{\alpha\in\overline{\Sigma}} \hat{u}_\alpha(x)t^\alpha.$$

Suppose $P\hat{u} \equiv f \mod \sum_{\alpha\in\partial\Sigma} \hat{\mathcal{A}}_M^m t^\alpha$. Put $h = f - P\bar{u}$. Then

$$h = \sum_{\alpha\in\mathbb{N}^n\setminus\overline{\Sigma}} h_\alpha(x)t^\alpha = \sum_{\alpha\in\mathbb{N}^n\setminus\overline{\Sigma},\ \beta\in\mathbb{N}^{n'}} h_{\alpha,\beta}t^\alpha x^\beta \in \mathcal{A}_M^m$$

and

$$P\hat{u} = f \iff Pu = h \quad \text{with } u = \hat{u} - \bar{u}.$$

Then the equation $P\hat{u} = f$ is equal to

$$\sigma_*(P)(x,\vartheta)u = h - \sum_{\substack{(\alpha,\beta)\in\mathbb{N}^{n+n'}\\ |\alpha|+|\beta|\le r}} p_{\alpha,\beta}(t,x)\vartheta^\alpha\partial_x^\beta u.$$

$$u = \sum_{\alpha\in\mathbb{N}^n} u_\alpha(x)t^\alpha \quad \text{with } u_\alpha(x) = \begin{cases} 0 & \text{for } \alpha\in\overline{\Sigma},\\ \hat{u}_\alpha(x) & \text{for } \alpha\in\mathbb{N}^n\setminus\overline{\Sigma}, \end{cases}$$

which also equals

$$\sigma_*(P)(x,\alpha^o)u_{\alpha^o}(x) = h_{\alpha^o}(x) \\ - \operatorname{Coef}(t^{\alpha^o}) \text{ of } \Big(\sum_{\substack{(\alpha,\beta)\in\mathbb{N}^{n+n'}\\ |\alpha|+|\beta|\le r}} p_{\alpha,\beta}(t,x)\vartheta^\alpha\partial_x^\beta\Big)\Big(\sum_{\substack{\alpha\in\mathbb{N}^n\\ |\alpha|<|\alpha^o|}} u_\alpha(x)t^\alpha\Big) \tag{6}$$

for $\forall\alpha^o \in \mathbb{N}^n \setminus \overline{\Sigma}$. Here "Coef($t^{\alpha^o}$)" means "the coefficient of t^{α^o}". Since $\det\sigma_*(P)(x,\gamma) \neq 0$ for $\gamma \in \mathbb{N}^n \setminus \overline{\Sigma}$, $u_{\alpha^o}(x)$ is inductively determined by h.

On the other hand, putting $h = 0$, it is clear that the claim i) follows from the induction proving $u_{\alpha^o} = 0$ by (6) for $\forall\alpha^o \in \mathbb{N}^n \setminus \Sigma$.

Put

$$u_\alpha(x) = \sum_{\beta\in\mathbb{N}^{n'}} u_{\alpha,\beta}x^\beta \quad \text{with} \quad u_{\alpha,\beta} \in \mathbb{C}.$$

The equation (6) equals

$$\sigma_*(P)(0,\alpha^o)u_{\alpha^o,\beta^o} \\ = h_{\alpha_o,\beta} + \operatorname{Coef}(x^{\beta^o}) \text{ of } \big(\sigma_*(P)(0,\alpha^o) - \sigma_*(P)(x,\alpha^o)\big)\Big(\sum_{|\beta|<|\beta^o|} u_{\alpha^o,\beta}x^\beta\Big) \\ - \operatorname{Coef}(t^{\alpha^o}x^{\beta^o}) \text{ of } \Big(\sum_{\substack{(\alpha,\beta)\in\mathbb{N}^{n+n'}\\ |\alpha|+|\beta|\le r}} p_{\alpha,\beta}(t,x)\vartheta^\alpha\partial_x^\beta\Big)\Big(\sum_{\substack{(\alpha,\beta)\in\mathbb{N}^{n+n'}\\ |\alpha|<|\alpha^o|}} u_{\alpha,\beta}t^\alpha x^\beta\Big)$$

for any $\alpha^o \in \mathbb{N}^n \setminus \overline{\Sigma}$ and $\beta^o \in \mathbb{N}^{n'}$. Hence the elements u_{α^o,β^o} of $\mathbb{C}^m$ satisfying this equation are uniquely and inductively determined in the lexicographic order of $(|\alpha^o|, |\beta^o|)$. Thus to complete the proof we have only to prove that $\sum u_{\alpha,\beta}t^\alpha x^\beta$ is a convergent power series. Here we may assume $\overline{\Sigma} \ni \{0\}$.

In general, for formal power series $\psi = \sum a_{\alpha,\beta}t^\alpha x^\beta$ and $\phi = \sum b_{\alpha,\beta}t^\alpha x^\beta$ we denote $\psi \ll \phi$ if $|a_{\alpha,\beta}| \le b_{\alpha,\beta}$ for $\forall\alpha$, β and in this case ϕ is called a majorant series of ψ. Note that if ϕ is a convergent power series, so is ψ.

Now assume (5). We note that there exists $\epsilon > 0$ such that

$$|\det\bar{\sigma}_*(P)(0,\xi)| \ge \epsilon(\xi_1 + \cdots + \xi_n)^{mr} \quad \text{for } \forall\xi \in [0,\infty)^n.$$

As in the proof of [O3, Theorem 2.1], we can choose $C > 0$, $c > 0$, $M > 0$ and $K \ge 1$ so that for $\forall(\alpha,\beta) \in \mathbb{N}^{n+n'}$ and $\forall\gamma \in \mathbb{N}^n \setminus \Sigma$

$$cm\big|\big(\sigma_*(P)(0,\gamma)^{-1}\big)_{ij}\big| \le \prod_{j=0}^{r-1}\big(r|\gamma| - j\big)^{-1},$$

$$\sigma_*(P)(x,\gamma)_{ij} - \sigma_*(P)(0,\gamma)_{ij} \le \frac{C(x_1 + \cdots + x_{n'})\prod_{j=0}^{r-1}\big(r|\gamma| - j\big)}{1 - K(x_1 + \cdots + x_{n'})},$$

$$p_{\alpha,\beta}(t,x)_{ij} - p_{\alpha,\beta}(0,x)_{ij} \ll \frac{C(t_1 + \cdots + t_n)}{1 - K(t_1 + \cdots + t_n + x_1 + \cdots + x_{n'})}$$

$$h(t,x)_i \ll \frac{M(t_1 + \cdots + t_n)}{1 - K(t_1 + \cdots + t_n + x_1 + \cdots + x_{n'})}.$$

Here i and j represent the indices of square matrices or vectors of size m. Hence the power series $w(s,y)$ of (s,y) satisfying

$$\begin{aligned} c\prod_{j=0}^{r-1}\Bigl(rs\frac{\partial}{\partial s}-j\Bigr)w &= \frac{Cmy}{1-Ky}\prod_{j=0}^{r-1}\Bigl(rs\frac{\partial}{\partial s}-j\Bigr)w \\ &\quad + \sum_{j+k\le r}\frac{Cm(n+n')^r s}{1-K(s+y)}\Bigl(s\frac{\partial}{\partial s}\Bigr)^j\Bigl(\frac{\partial}{\partial y}\Bigr)^k w \\ &\quad + \frac{Ms}{1-K(s+y)}, \\ w(0,y) &= 0 \end{aligned} \tag{7}$$

implies

$$\Bigl(u(t,x)-\sum_{\alpha\in\mathbb{N}^n\setminus\overline{\Sigma}}u_\alpha(x)t^\alpha\Bigr)_i \ll w(t_1+\cdots+t_n, x_1+\cdots+x_{n'}) \quad \text{for } 1\le i\le m.$$

Put $s=z^r$. Then (7) changes into

$$\begin{aligned} \Bigl(c-\frac{Cmy}{1-Ky}\Bigr)z^r\frac{\partial^r w}{\partial z^r} &= \sum_{j+k\le r}\frac{Cm(n+n')^r z^r}{1-K(z^r+y)}\Bigl(\frac{z}{r}\frac{\partial}{\partial z}\Bigr)^j\frac{\partial^k w}{\partial y^k} \\ &\quad + \frac{Mz^r}{1-K(z^r+y)}, \\ \frac{\partial^j w}{\partial z^j}\Big|_{z=0} &= 0 \quad \text{for} \quad j=0,\dots,r-1. \end{aligned} \tag{8}$$

Since the first equation in the above is equivalent to

$$\Bigl(c-\frac{Cmy}{1-Ky}\Bigr)\frac{\partial^r w}{\partial z^r} = \sum_{j+k\le r}\frac{Cm(n+n')^r}{1-K(z^r+y)}\Bigl(\frac{z}{r}\frac{\partial}{\partial z}\Bigr)^j\frac{\partial^k w}{\partial y^k} + \frac{M}{1-K(z^r+y)},$$

(8) has a unique solution of power series of (y,z), which is assured to be analytic at the origin by Cauchy-Kowalevsky's theorem. In fact for a sufficiently large positive number L, the solution of the ordinary differential equation

$$\begin{aligned} \Bigl(c-\frac{Cmt}{1-Kt}\Bigr)\tilde{w}^{(r)}(t) &= \sum_{j+k\le r}\frac{Cm(n+n')^r L^{-k}}{1-Kt}\Bigl(\frac{t}{r}\frac{\mathrm{d}}{\mathrm{d}t}\Bigr)^j\tilde{w}^{(k)}(t)+\frac{M}{1-Kt}, \\ \tilde{w}^{(j)}(0) &= 0 \quad \text{for} \quad j=0,\dots,r-1 \end{aligned}$$

with

$$\begin{cases} t = z+Ly, \\ cL^r > Cm(n+n')^r \end{cases}$$

satisfies $w(z,y)\ll\tilde{w}(z+Ly)$. Hence u is also a convergent power series. □

Let ℓ be a non-negative integer and let U be an open connected neighborhood of a point z^o of $\mathbb{C}^\ell$ and let $\mathcal{O}_U$ be the space of holomorphic functions on U. We denote by ${}_U\mathcal{A}_M$ and ${}_U\mathcal{A}_N$ the space of real analytic functions on M with holomorphic parameter $z \in U$ and that on N with holomorphic parameter $z \in U$, respectively. Moreover we denote by ${}_U\hat{\mathcal{A}}_M$ the space of formal power series of $t = (t_1, \dots, t_n)$ with coefficients in ${}_U A_N$. Let ${}_U\mathcal{D}_*$ denote the ring of differential operators P of the form

$$\begin{cases} P = \sum_{(\alpha,\beta)\in\mathbb{N}^{n+n'}} a_{\alpha,\beta}(t,x,z)\vartheta^\alpha \partial_x^\beta, \\ a_{\alpha,\beta} \in {}_U\mathcal{A}_N,\ a_{\alpha,\beta}(0,x,z) = 0 \quad \text{if} \quad \beta > 0. \end{cases}$$

Then $\sigma_*(P)(x,z,\xi) := \sum_\alpha p_{\alpha,0}(0,x,z)\xi^\alpha \in {}_U\mathcal{A}_N[\xi]$.

Theorem 3.2. *Let $P \in M(m, {}_U\mathcal{D}_*)$ and $\lambda(z) = \big(\lambda_1(z), \dots, \lambda_n(z)\big) \in \mathcal{O}_U^n$.*

i) *Let Σ be a subset of $\mathbb{N}^n$ such that*

$$\det\big(\sigma_*(P)(x,z,\lambda(z)+\gamma)\big) \not\equiv 0 \quad \textit{for} \quad \forall\gamma \in \mathbb{N}^n \setminus \Sigma.$$

Let $\phi(t,x,z) = \sum_{\alpha\in\mathbb{N}^n} \phi_\alpha(x,z)t^\alpha \in {}_U\hat{\mathcal{A}}_M^m$ satisfying $P\big(t^{\lambda(z)}\phi\big) = 0$. Then $\phi = 0$ if $\phi_\alpha = 0$ for $\forall\alpha \in \Sigma$.

Hereafter in this theorem suppose P satisfies

$$\det \bar\sigma_*(P)(x,z,\xi) \neq 0 \quad \textit{for}\ \forall(x,z,\xi) \in N \times U \times \big\{[0,\infty)^n \setminus \{0\}\big\}. \tag{9}$$

ii) *If $\phi(t,x,z) \in {}_U\hat{\mathcal{A}}_M^m$ satisfies $P\big(t^{\lambda(z)}\phi\big) = 0$, then $\phi \in {}_U\mathcal{A}_M^m$.*

iii) *Fix $x^o \in N$. Let Σ be a finite subset Σ of $\mathbb{N}^n$ such that*

$$\det\big(\sigma_*(P)(x^o,z^o,\lambda(z^o)+\gamma)\big) \neq 0 \quad \textit{for} \quad \forall\gamma \in \mathbb{N}^n \setminus \Sigma.$$

Shrinking U and N if necessary and denoting

$$\begin{aligned} \mathrm{Sol}_U(P;\lambda) :&= \{u\,;\, ut^{-\lambda(z)} \in {}_U\mathcal{A}_M^m \textit{ and } Pu = 0\}, \\ \mathrm{Sol}_U(P;\lambda)^\Sigma :&= \{\bar u = \sum_{\alpha\in\overline{\Sigma}} \phi_\alpha(x,z)t^{\lambda(z)+\alpha}\,;\, \bar u t^{-\lambda(z)} \in {}_U\mathcal{A}_M^m \textit{ and} \\ &\qquad P\bar u \equiv 0 \mod \sum_{\beta\in\partial\Sigma} {}_U\mathcal{A}_M^m t^{\lambda(z)+\beta}\}, \end{aligned}$$

we see that the natural restriction map

$$\begin{gathered} \mathrm{Sol}_U(P;\lambda) \xrightarrow{\sim} \mathrm{Sol}_U(P;\lambda)^\Sigma, \\ \sum_{\alpha\in\mathbb{N}^n} \phi_\alpha(x,z)t^{\lambda(z)+\alpha} \mapsto \sum_{\alpha\in\Sigma} \phi_\alpha(x,z)t^{\lambda(z)+\alpha} \end{gathered}$$

is a bijection. Here in particular

$$\mathrm{Sol}_U(P;\lambda)^{\{0\}} = \{u \in {}_U\mathcal{A}_N^m\,;\, \sigma_*(P)\big(x,z,\lambda(z)\big)u = 0\}.$$

Proof. Fix $x^o \in N$. Expanding functions in convergent power series of (t,x,z) at $(0,x^o,z^o)$, we will prove the lemma in a neighborhood of $(0,x^o,z^o)$. Replacing P and the complexification $M_{\mathbb{C}}$ of M by $t^{-\lambda(z)} \circ P \circ t^{\lambda(z)}$ and $M_{\mathbb{C}} \times U$, respectively, we can reduce this theorem to the previous theorem without the parameter z. □

Corollary 3.3. *Retain the notation in the previous theorem. Let $\ell = 1$. Suppose*

$$\sigma_*(P)(x,z,\lambda(z)) = 0 \quad \textit{for} \ \ \forall (x,z) \in N \times U$$

and

$$\det\bigl(\sigma_*(P)(x^o,z,\lambda(z)+\gamma)\bigr) \neq 0 \quad \textit{for} \ \ \forall \gamma \in \mathbb{N}^n \setminus \{0\} \ \textit{and} \ \forall z \in U \setminus \{z^o\}.$$

Then there exists a non-negative integer k such that the following holds.

The previous theorem assures that for any $\phi_0(x,z) \in {}_U\mathcal{A}_N^m$ and fixed $z \in U \setminus \{z^o\}$ there exists a function $u(t,x,z)$ satisfying

$$\begin{cases} Pu = 0, \\ t^{-\lambda(z)} u \in \mathcal{A}_M^m, \\ t^{-\lambda(z)} u|_{t=0} = \phi_0(x,z). \end{cases}$$

Then $t^{-\lambda(z)} z^k u(x,z)$ extends holomorphically to the point $z = z^o$.

Proof. Since the functions $\det\bigl(\sigma_*(P)(x^o,z,\lambda(z)+\gamma)\bigr)$ have finite order of zeros at $z = z^0$ for $\gamma \in \Sigma \setminus \{0\}$, this corollary follows from the proof of Theorem 3.1 (cf. (6) for $\forall \alpha^o \in \mathbb{N}^n \setminus \{0\}$). In fact it is sufficient to put k the sum of these orders of zeros for $\gamma \in \Sigma \setminus \{0\}$. □

Remark 3.4. It follows from the proves of Theorem 3.1 and Theorem 3.2 that there exist differential operators $P_\alpha^\gamma(x,z,\partial_x)$ such that

$$\phi_\alpha(x,z) = \sum_{\gamma \in \Sigma} P_\alpha^\gamma(x,z,\partial_x)\phi_\gamma(x,z) \quad \text{for } \alpha \in \mathbb{N}^n \setminus \Sigma$$

in Theorem 3.2 iii).

Corollary 3.5. *Fix $(x^o,\lambda^o) \in N \times \mathbb{C}^n$ and let V be a neighborhood of λ^o in $\mathbb{C}^n$. Suppose $P \in M(m,\mathcal{D}_*)$ satisfies* (9) *and*

$$\det\bigl(\sigma_*(P)(x^o,\lambda^o+\gamma) - \sigma_*(P)(x^o,\lambda^o)\bigr) \neq 0 \quad \textit{for} \ \ \forall \gamma \in \mathbb{N}^n \setminus \{0\}.$$

Then shrinking N, M and V if necessary, we have a linear bijection

$$\begin{aligned} \beta_\lambda : \mathrm{Sol}_V(P) := \{u\,;\, ut^{-\lambda} \in {}_V\mathcal{A}_M^m \ \textit{and} \ Pu = \sigma_*(P)(x,\lambda)u\} &\xrightarrow{\sim} {}_V\mathcal{A}_N^m, \\ u &\mapsto t^{-\lambda} u|_{t=0} \end{aligned}$$

with the coordinate $((t,x),\lambda) \in M \times V$. In particular, we have a bijective map

$$\begin{aligned} \beta_{\lambda^o} : \mathrm{Sol}_{\lambda^o}(P) := \{u\,;\, ut^{-\lambda^o} \in \mathcal{A}_M^m \ \textit{and} \ Pu = \sigma_*(P)(x,\lambda^o)u\} &\xrightarrow{\sim} \mathcal{A}_N^m, \\ u &\mapsto t^{-\lambda^o} u|_{t=0}. \end{aligned}$$

Definition 3.6. *The map* β_{λ^o} *of* $\mathrm{Sol}_{\lambda^o}(P)$ *is called the* boundary value map *of the solution space* $\mathrm{Sol}_{\lambda^o}(P)$ *of the differential equation* $Pu = \sigma_*(P)(x, \lambda^o)u$ *with respect to the* characteristic exponent λ^o.

Remark 3.7. When $n = 1$, $u \in \mathrm{Sol}_{\lambda^o}(P)$ is called an *ideally analytic solution* of the equation $Pu = \sigma_*(P)(x, \lambda^o)u$ in [KO].

The following theorem says that $\mathrm{Sol}_V(P)$ and $\sigma_*(P)$ characterize $P \in \mathcal{D}_*$.

Theorem 3.8. *Let* P *be an element of* $M(m, \mathcal{D}_*)$ *satisfying the assumptions in Corollary 3.5. Let* $P' \in M(m, \mathcal{D}_*)$ *with* $\sigma_*(P) = \sigma_*(P')$*. Then the condition* $\mathrm{Sol}_V(P) = \mathrm{Sol}_V(P')$ *implies* $P = P'$.

Proof. Suppose $P \neq P'$. Put

$$P - P' = \sum_{\alpha,\beta,\gamma} r_{\alpha,\beta,\gamma} t^\gamma \vartheta^\alpha \partial_x^\beta.$$

Then we can find $\gamma^o \in \mathbb{N}^{n'} \setminus \{0\}$ such that $\sum_{\alpha,\beta} r_{\alpha,\beta,\gamma^o} t^{\gamma^o} \vartheta^\alpha \partial_x^\beta \neq 0$ and $r_{\alpha,\beta,\gamma} = 0$ if $\gamma < \gamma^o$. For $v(x) \in \mathcal{A}_N^m$ the coefficients of $t^{\lambda+\gamma^o}$ in $(P-P')\beta_\lambda^{-1} v(x)$ show

$$\begin{aligned} 0 &= \big(t^{-\lambda} \sum_{\alpha,\beta} r_{\alpha,\beta,\gamma^o} \vartheta^\alpha \partial_x^\beta t^\lambda v(x)\big)|_{t=0} \\ &= \sum_{\alpha,\beta} r_{\alpha,\beta,\gamma^o} \lambda^\alpha \partial_x^\beta v(x) \quad \text{for} \quad \forall\lambda \in V \text{ and } \forall v(x) \in \mathcal{A}_N^m, \end{aligned}$$

which means a contradiction. □

4 Induced equations

Retain the notation in the previous section. Moreover we denote by ${}_U\widetilde{\mathcal{D}}_*$ the ring of holomorphic maps of U to $\widetilde{\mathcal{D}}_*$ for a connected open subset U of $\mathbb{C}^\ell$.

We recall that the element P of ${}_U\widetilde{\mathcal{D}}_*$ is characterized by the expression

$$P = \sum_{(\alpha,\beta)\in\mathbb{N}^{n+n'}} p_{\alpha,\beta}(t, x, z) \vartheta^\alpha \partial_x^\beta \tag{10}$$

with $p_{\alpha,\beta}(t, x, z) \in {}_U\mathcal{A}_M$ and

$$\sigma_*(P)(x, z, \xi, \partial_x) = \sum_{\alpha,\beta} p_{\alpha,\beta}(0, x, z) \xi^\alpha \partial_x^\beta.$$

Theorem 4.1. *Let* $P \in M(m, {}_U\mathcal{D}_*)$ *satisfying the assumption in Theorem 3.2 iii) with* $\Sigma = \{0\}$*. Suppose that* $P_1, \ldots, P_p \in M(m, {}_U\widetilde{\mathcal{D}}_*)$ *satisfy*

$$[P, P_i] = S_i P + \sum_{j=1}^{p} T_{ij} P_j \tag{11}$$

with $S_i \in M(m, {}_U\widetilde{\mathcal{D}}_*)$ *and* $T_{ij} \in M(m, {}_U\mathcal{D}_*)$. *Suppose moreover* $\sigma_*(T_{ij}) = 0$. *Then the map*

$$\begin{aligned}\beta_{\lambda(z)} :&\{u\,;\, t^{-\lambda(z)}u \in {}_U\mathcal{A}_M^m \text{ and } Pu = P_i u = 0 \text{ for } i = 1,\dots,p\}\\ &\xrightarrow{\sim} \Bigl\{v \in {}_U\mathcal{A}_N^m\,;\, \begin{cases}\sigma_*(P)\bigl(x,z,\lambda(z)\bigr)v = 0,\\ \sigma_*(P_i)\bigl(x,z,\lambda(z),\partial_x\bigr)v = 0 \quad (i = 1,\dots,p)\end{cases}\Bigr\}, \\ &u \mapsto t^{-\lambda(z)}u\,\bigr|_{t=0}\end{aligned} \tag{12}$$

is a bijection.

Proof. Since $(t^{-\lambda(z)}P_j u)|_{t=0} = \sigma_*(P_j)(x,z,\lambda(z),\partial_x)t^{-\lambda(z)}u|_{t=0}$, Theorem 3.2 assures that we have only to prove the surjectivity of the map to get the theorem.

For a given v in the element of the set, we have $u \in t^{\lambda(z)}{}_U\mathcal{A}_M^m$ such that $Pu = 0$ and $t^{-\lambda(z)}u|_{t=0} = v$. Then $PP_i u = \sum_{j=1}^p T_{ij}P_j u$, namely,

$$\begin{pmatrix} P - T_{11} & -T_{12} & -T_{13} & \cdots & -T_{1p}\\ -T_{21} & P - T_{22} & -T_{23} & \cdots & -T_{2p}\\ -T_{31} & -T_{32} & P - T_{33} & \cdots & -T_{2p}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -T_{p1} & T_{p2} & T_{p3} & \cdots & P - T_{pp}\end{pmatrix}\begin{pmatrix}P_1u\\ P_2u\\ P_3u\\ \vdots\\ P_pu\end{pmatrix} = 0.$$

Since $\sigma_*(T_{ij}) = 0$ and $t^{-\lambda(z)}P_j u|_{t=0} = 0$ for $j = 1,\dots,p$, Theorem 3.2 i) assures $P_j u = 0$. □

Definition 4.2. *The system of differential equations*

$$\sigma_*(P)(x,z,\lambda(z))v = \sigma_*(P_i)(x,z,\lambda(z),\partial_x)v = 0 \quad \textit{for} \ \ i = 1,\dots,p$$

in Theorem 4.1 is called the system of induced equations *with respect to the boundary value map* $\beta_{\lambda(z)}$ (cf. (12)).

Remark 4.3. i) Suppose $P \in M(m, {}_U\mathcal{D}_*)$ satisfies the assumption in Theorem 4.1. Let $Q \in M(m, {}_U\mathcal{D}_*)$ such that $[P,Q] = 0$ and $\sigma_*(Q)\bigl(x,z,\lambda(z)\bigr) = 0$. Then if $u \in t^{\lambda(z)}{}_U\mathcal{A}_M^m$ satisfies $Pu = 0$, we have $Qu = 0$.

ii) Let p be the rank of an irreducible semisimple symmetric space G/H. The ring of invariant differential operators on G/H is isomorphic to $\mathbb{C}[P_1,\dots,P_p]$, where P_j are algebraically independent and satisfy $[P_i,P_j] = 0$ for $1 \le i < j \le p$. Under a suitable coordinate system $(t_1,\dots,t_n,x_1,\dots,x_{n'})$ of a natural realization of G/H constructed by [O6], G/H is defined by $t_1 > 0,\dots,t_n > 0$. Then n is the real rank of G/H and $P_i \in \tilde{\mathcal{D}}_* \setminus \mathcal{D}_*$ if $n < p$. It is shown in [O6] that we can choose $P \in \sum_{j=1}^p \mathcal{D}_* P_j$ such that P, $P_1,\dots,P_p$ satisfy the assumption in Theorem 4.1.

5 Holonomic systems of differential equations with constant coefficients

In this section $\bigl(\frac{\partial}{\partial y_1},\dots,\frac{\partial}{\partial y_n}\bigr)$ is simply denoted by ∂. For $\mu=(\mu_1,\dots,\mu_n)\in\mathbb{C}^n$ and $y=(y_1,\dots,y_n)\in\mathbb{R}^n$, we put

$$\langle\mu,y\rangle=\mu_1y_1+\cdots+\mu_ny_n.$$

Lemma 5.1. *Let* $\operatorname{Hom}_{\mathbb{C}[\partial]}(\mathcal{M},\mathcal{N})$ *denote the space of* $\mathbb{C}[\partial]$*-homomorphisms of a* $\mathbb{C}[\partial]$*-module* $\mathcal{M}$ *to a* $\mathbb{C}[\partial]$*-module* $\mathcal{N}$*. Then the space is naturally a* $\mathbb{C}[\partial]$*-module. Let* $\hat{\mathcal{O}}$ *be the space of formal power series of* $y=(y_1,\dots,y_n)$ *and let* $\mathcal{O}(\mathbb{C}^n)$ *be the space of entire functions on* $\mathbb{C}^n\ni y$*. Suppose* $\mathcal{M}$ *is a finite dimensional* $\mathbb{C}[\partial]$*-module. Then*

$$\begin{aligned}\bigoplus_{\lambda\in\mathbb{C}^n}\operatorname{Hom}_{\mathbb{C}[\partial]}\bigl(\mathcal{M},\mathbb{C}[y]e^{\langle\lambda,y\rangle}\bigr)&\xrightarrow{\sim}\operatorname{Hom}_{\mathbb{C}[\partial]}\bigl(\mathcal{M},\bigoplus_{\lambda\in\mathbb{C}^n}\mathbb{C}[y]e^{\langle\lambda,y\rangle}\bigr)\\&\xrightarrow{\sim}\operatorname{Hom}_{\mathbb{C}[\partial]}\bigl(\mathcal{M},\mathcal{O}(\mathbb{C}^n)\bigr)\\&\xrightarrow{\sim}\operatorname{Hom}_{\mathbb{C}[\partial]}\bigl(\mathcal{M},\hat{\mathcal{O}}\bigr),\end{aligned}\tag{13}$$

$$\dim\operatorname{Hom}_{\mathbb{C}[\partial]}\bigl(\mathcal{M},\mathcal{O}(\mathbb{C}^n)\bigr)=\dim\mathcal{M}.\tag{14}$$

If $\mathcal{M}'$ *is a quotient* $\mathbb{C}[\partial]$*-module of* $\mathcal{M}$ *such that*

$$\operatorname{Hom}_{\mathbb{C}[\partial]}\bigl(\mathcal{M}',\mathcal{O}(\mathbb{C}^n)\bigr)\xrightarrow{\sim}\operatorname{Hom}_{\mathbb{C}[\partial]}\bigl(\mathcal{M},\mathcal{O}(\mathbb{C}^n)\bigr),$$

then $\mathcal{M}\xrightarrow{\sim}\mathcal{M}'$*.*

Proof. For $\mu=(\mu_1,\dots,\mu_n)\in\mathbb{C}^n$, let $\mathfrak{m}_\mu$ denote the maximal ideal of $\mathbb{C}[\partial]$ generated by $\frac{\partial}{\partial y_i}-\mu_i$ with $i=1,\dots,n$. Then we have $\mathcal{M}\simeq\mathcal{M}_{\lambda_1}\oplus\cdots\oplus\mathcal{M}_{\lambda_m}$ with suitable $\lambda_\nu=(\lambda_{\nu,1},\dots,\lambda_{\nu,n})\in\mathbb{C}^n$ and $\mathbb{C}[\partial]$-modules $\mathcal{M}_{\lambda_\nu}$ satisfying $\mathfrak{m}_{\lambda_\nu}^k\mathcal{M}_{\lambda_\nu}=0$ for a large positive integer k. Hence we have only to prove the lemma for each $\mathcal{M}_{\lambda_\nu}$. By the outer automorphism $\frac{\partial}{\partial y_i}\mapsto\frac{\partial}{\partial y_i}+\lambda_{\nu,i}$ for $i=1,\dots,n$ which corresponds to the multiplication of the functions in $\mathcal{O}(\mathbb{C}^n)$ or $\hat{\mathcal{O}}$ by $e^{-\langle\lambda_\nu,x\rangle}$ we may assume $\mathfrak{m}_0^k\mathcal{M}=0$.

Suppose $\mathfrak{m}_0^k\mathcal{M}=0$. Then $\operatorname{Hom}_{\mathbb{C}[\partial]}\bigl(\mathcal{M},\mathbb{C}[y]\bigr)\xrightarrow{\sim}\operatorname{Hom}_{\mathbb{C}[\partial]}\bigl(\mathcal{M},\hat{\mathcal{O}}\bigr)$ and (13) is clear. Since $\hat{\mathcal{O}}$ is the dual space of $\mathbb{C}[\partial]$ by the bilinear form $\langle P(\partial),u\rangle=P(\partial)u|_{x=0}$, (14) is clear. The last statement follows from (14). □

Definition 5.2. *A finite dimensional* $\mathbb{C}[\partial]$*-module* $\mathcal{M}$ *is* semisimple *if*

$$\operatorname{Hom}_{\mathbb{C}[\partial]}\bigl(\mathcal{M},\bigoplus_{\lambda\in\mathbb{C}^n}\mathbb{C}e^{\langle\lambda,y\rangle}\bigr)\xrightarrow{\sim}\operatorname{Hom}_{\mathbb{C}[\partial]}\bigl(\mathcal{M},\mathcal{O}(\mathbb{C}^n)\bigr).$$

Let U be a convex open subset of $\mathbb{C}^\ell$, where ℓ is a non-negative integer, and let ${}_U\mathbb{C}[\partial]$ and ${}_U\mathcal{O}(\mathbb{C}^n)$ be the space of holomorphic maps of U to $\mathbb{C}[\partial]$ and that of U to $\mathcal{O}(\mathbb{C}^n)$, respectively.

Proposition 5.3. *Let r be a positive integer and let ${}_U\mathcal{M}$ be a finitely generated ${}_U\mathbb{C}[\partial]$ module with $\dim {}_U\mathcal{M} = r$ for any fixed $z \in U$. Assume that there exist positive integer k and finite number of holomorphic maps λ_i of U to $\mathbb{C}^n$ such that $\bigl(\prod_{i\in I}\mathfrak{m}^k_{\lambda_i(z)}\bigr){}_U\mathcal{M} = 0$ for any $z \in U$. Here the indices i run over a finite set I. Then there exist ${}_U\mathbb{C}[\partial]$-homomorphisms $u_1,\dots,u_r$ of ${}_U\mathcal{M}$ to ${}_U\mathcal{O}(\mathbb{C}^n)$ such that they are linearly independent for any fixed $z \in U$.*

Let $I = I_1 \cup \cdots \cup I_L$ be a decomposition of I such that

$$\lambda_i(z) \neq \lambda_j(z) \quad \text{for } \forall z \in U \quad \text{if } i \in I_\mu \text{ and } j \in I_\nu \quad \text{and} \quad 1 \le \mu < \nu \le L.$$

Then we can choose $\{u_i\,;\, i \in I\}$ such that for each u_i there exists I_ν satisfying

$$u_i \in \operatorname{Hom}_{{}_U\mathbb{C}[\partial]}\bigl({}_U\mathcal{M}, \sum_{j\in I_\nu} \mathrm{e}^{\langle\lambda_j(z),y\rangle}\mathbb{C}[y]\bigr) \quad \text{for any fixed } z \in U. \tag{15}$$

Proof. Let $\{v_1,\dots,v_m\}$ be a system of generators of ${}_U\mathcal{M}$. We identify the homomorphisms of ${}_U\mathcal{M}$ to ${}_U\mathcal{O}(\mathbb{C}^n)$ with their image of $\{v_1,\dots,v_m\}$ and hence $u_j(y,z) \in {}_U\mathcal{O}(\mathbb{C}^n)^m$. Note that we can find ${}_U\mathbb{C}[\partial]$-homomorphisms $\tilde u_1(y,z),\dots,\tilde u_r(y,z)$ of ${}_U\mathcal{M}$ to ${}_U\mathcal{O}(\mathbb{C}^n)$ if we replace $\mathcal{O}(U)$ by its quotient field.

Fix a point $z^o \in U$. Let $\gamma(t)$ be a holomorphic map of $\{t \in \mathbb{C}\,;\, |t| < 1\}$ to U such that $\gamma(0) = z^o$ and $\tilde u_j(y,\gamma(t))$ are holomorphic and linearly independent for $0 < |t| < 1$. Then [OS, Proposition 2.21] assures that there exist meromorphic functions $c_{ij}(t)$ such that the functions $v_i(y,t) = \sum_{j=1}^r c_{ij}(t)\tilde u_j(y,\gamma(t))$ are holomorphic at $t = 0$ and that $v_1(y,0),\dots,v_r(y,0)$ are linearly independent. We can find $P_i \in \mathbb{C}[\partial]^m$ such that $\langle P_i, v_j\rangle = \delta_{ij}$ for $1 \le i \le r$ and $1 \le j \le r$. Here we put $\langle (Q_1,\dots,Q_m),(f_1,\dots,f_m)\rangle := \sum_{\nu=1}^m Q_\nu(f_\nu)(0)$ for $Q_\nu \in \mathbb{C}[\partial]^m$ and $f_\nu \in \mathcal{O}(\mathbb{C}^n)^m$.

Put $A(z) = \Bigl(\langle P_i, \tilde u_j\rangle\Bigr)_{\substack{1\le i\le r\\ 1\le j\le r}}$, which is a matrix of meromorphic functions on U and $\det A(z)$ is not identically zero. Let $\tilde c_{ij}(z)$ are meromorphic functions on U such that $\langle P_i, u_j\rangle = \delta_{ij}$ by putting $u_i = \sum_{j=1}^r \tilde c_{ij}(z)\tilde u_j$.

Suppose $u_i(y,z)$ is not holomorphic at $z = z^0$. Then there exist a positive integer L and a holomorphic function $\tilde\gamma$ of $\{t \in \mathbb{C}\,;\, |t| < 1\}$ to U such that $\tilde\gamma(0) = z^0$ and the function $w(y,t) := t^L u_i(y,\tilde\gamma(t))$ is holomorphically extended to the point $t = 0$ and moreover $w(y,0) \neq 0$. Then $w(y,0)$ defines a $\mathbb{C}[\partial]$-homomorphism of ${}_U\mathcal{M}$ to $\mathcal{O}(\mathbb{C}^n)$ at $z = z^o$. But $w(y,0), v_1(y,0),\dots,v_r(y,0)$ are linearly independent because $\langle P_i, w(y,0)\rangle = 0$ for $i = 1,\dots,r$, which contradicts to (14).

Hence for any $z^o \in U$ we can construct $u_1(y,z),\dots,u_r(y,z)$ which are linearly independent and holomorphic in a neighborhood of $z^o \in U$. Then the theorem follows from the theory of holomorphic functions with several variables because U is a convex open subset of $\mathbb{C}^\ell$.

Since we have a decomposition ${}_U\mathcal{M} = {}_U\mathcal{M}_1 \oplus \cdots \oplus {}_U\mathcal{M}_L$ such that $\bigl(\prod_{i\in I_\nu}\mathfrak{m}^k_{\lambda_i(z)}\bigr){}_U\mathcal{M}_\nu = 0$ for $\nu = 1,\dots,L$, we can assume (15). □

Example 5.4. Let W be a finite reflection group on a Euclidean space $\mathbb{R}^n$. Let $\mathbb{C}[p_1,\dots,p_n]$ be the algebra of W-invariant polynomials on $\mathbb{R}^n$. For example, $p_k(x)=\sum_{1\le i_1<\cdots<i_k\le n} x_{i_1}\cdots x_{i_k}$. Then the system of differential equations

$$\mathcal{M}_\lambda : p_i(\partial)u = p_i(\lambda)u \quad \text{for} \quad i=1,\dots,n$$

with $\lambda\in\mathbb{C}^n$ is a fundamental example of a ${}_U\mathbb{C}[\partial]$-module in Proposition 5.3. Here $U=\mathbb{C}^n\ni\lambda$ and $r=\#W$. The system is semisimple if and only if $w\lambda\ne\lambda$ for $\forall w\in W\setminus\{e\}$. When $\lambda=0$, the solutions of this system are called harmonic polynomials for W. In this case, an explicit construction of solutions is given by [O5] such that $u_1(\lambda,y),\dots,u_r(\lambda,y)$ are entire functions of $(\lambda,y)\in\mathbb{C}^{2n}$ and linearly independent for any fixed $\lambda\in\mathbb{C}^n$.

Remark 5.5. We will apply the result in this section to our original systems with the coordinates $t_i=\mathrm{e}^{-y_i}$ for $i=1,\dots,n$. Then $\mathbb{C}[\partial]$ and $\mathrm{e}^{\langle\lambda,y\rangle}f(y)$ change into $\mathbb{C}[\vartheta]$ and $t^{-\lambda}f\bigl(-\log t_1,\dots,-\log t_n\bigr)$, respectively.

6 Ideally analytic solutions for complete systems

In this section we will study the system of differential equations

$$\mathcal{M} : P_i u = 0 \quad \text{for } i=0,1,\dots,q \tag{16}$$

with $P_i\in M(m,{}_U\mathcal{D}_*)$. Here $z\in U$ is a holomorphic parameter and U is a convex open subset of $\mathbb{C}^\ell$. We assume that $\sigma_*(P_i)$ do not depend on $x\in N$. We moreover assume that $P=P_0$ satisfies (5) and the system

$$\overline{\mathcal{M}} : \sigma_*(P_i)(z,\vartheta)\bar u = 0 \quad \text{for} \quad i=0,1,\dots,q, \tag{17}$$

which we call *indicial equation*, satisfies the assumption of Proposition 5.3. Then we call $\mathcal{M}$ a *complete system of differential equations with regular singularities along the set of walls* $\{N_1,\dots,N_n\}$.

For a non-negative integer k let $\mathbb{C}[\log t]_{(k)}$ denote the polynomial function of $(\log t_1,\dots,\log t_n)$ with degree at most k. Put $\mathbb{C}[\log t]=\bigcup_{k=1}^\infty\mathbb{C}[\log t]_{(k)}$.

Definition 6.1. *A solution $u(t,x,z)$ of $\mathcal{M}$ with the holomorphic parameter z is called an* ideally analytic solution *if $u(t,x,z)\in\bigoplus_{\lambda\in\mathbb{C}} t^\lambda\mathbb{C}[\log t]\mathcal{A}_M^m$ for any fixed $z\in U$.*

First we will examine the system $\mathcal{M}$ without the holomorphic parameter z or U is a point. Then let $\{\bar u_i = t^{\lambda_i}v_i(\log t)\,;\, i=1,\dots,r\}$ be a basis of the solutions of (17). Here $v_i(\xi)\in\mathbb{C}[\xi]$ and these λ_i are called *exponents* of the system $\mathcal{M}$. We define

$$\begin{cases} e(\bar u_i) := \lambda_i, \\ \deg(\bar u_i) := \deg v_i. \end{cases}$$

We may assume that for any $\lambda \in \mathbb{C}^n$ and $k \in \mathbb{N}$

$$\{\bar{u}_i\,;\,\bigl(e(\bar{u}_i), \deg(\bar{u}_i)\bigr) = (\lambda, k)\} \text{ is empty}$$
$$\text{or linearly independent in the space } t^\lambda \mathbb{C}[\log t]^m_{(k)}/t^\lambda \mathbb{C}[\log t]^m_{(k-1)}.$$

Definition 6.2. *Let $u(t,x)$ be an ideally analytic solution of $\mathcal{M}$. Then a non-zero function*

$$w(t,x) = \sum_\nu t^\lambda p_\nu(\log t)\phi_\nu(x) \tag{18}$$

with suitable $\lambda \in \mathbb{C}^n$, $p_\nu(\xi) \in \mathbb{C}[\xi]$ and $\phi_\nu(x) \in \mathcal{A}^m_N$ is called a leading term *of $u(t,x)$ if*

$$u(t,x) - w(t,x) \in \sum_{\substack{\mu\in\mathbb{C}^n\\ \lambda-\mu\notin\mathbb{N}^n}} t^\mu \mathbb{C}[\log t]\mathcal{A}^m_M$$

and λ is called a leading exponent *of this leading term. If $\{w_1(t,x), \dots, w_k(t,x)\}$ is the complete set of the leading terms of $u(t,x)$, we say $\sum_{i=1}^k w_i(t,x)$ the* complete leading term *of $u(t,x)$.*

Then we have the following theorem.

Theorem 6.3. *The leading term* (18) *of an ideally analytic solution $u(t,x)$ of $\mathcal{M}$ is a solution of* (17)*. Hence there exist $\phi_i(x) \in \mathcal{A}_M$ such that*

$$w(t,x) = \sum_{\lambda_i=\lambda} \bar{u}_i(t)\phi_i(x). \tag{19}$$

In particular, λ is an exponent of $\mathcal{M}$.

Assume

$$\det \sigma_*(P_1)(e(\bar{u}_i) + \gamma) \neq 0 \quad \text{for} \quad \gamma \in \mathbb{N}^n \setminus \{0\}. \tag{20}$$

Then for any $\phi(x) \in \mathcal{A}_N$ there exists a unique solution of $\mathcal{M}$ in the space $t^{e(\bar{u}_i)}\mathbb{C}[\log t]\mathcal{A}^m_M$ whose leading term equals $\phi(x)\bar{u}_i$. Denoting the solution by $T_{\bar{u}_i}(\phi)$, we have the following bijective isomorphism if (20) *is valid for $1 \leq i \leq r$.*

$$\mathcal{A}^r_N \xrightarrow{\sim} \{\textit{ideally analytic solutions of } \mathcal{M}\},\ (\phi_i) \mapsto \sum_{i=1}^r T_{\bar{u}_i}(\phi_i). \tag{21}$$

Proof. Examining the equation $Pu(t,x) = 0$ modulo $\sum_{\substack{\mu\in\mathbb{C}^n\\ \lambda-\mu\notin\mathbb{N}^n}} t^\mu \mathbb{C}[\log t]\mathcal{A}^m_M$, we have $\sigma_*(P)(\vartheta)w(t,x) = 0$ and thus (19).

Put $\lambda = e(\bar{u}_i)$. First suppose $\deg(\bar{u}_i) = 0$. Then under the condition (20), Theorem 3.1 assures the unique existence of $\tilde{\phi}(t,x) \in \mathcal{A}^m_M$ such that $P_1 t^\lambda \tilde{\phi}(t,x) = 0$ and $t^{e(\bar{u}_i)}\tilde{\phi}(0,x) = \phi(x)u_i(t)$ and moreover Theorem 4.1 assures $P_j t^\lambda \tilde{\phi}(t,x) = 0$. If there exists another solution $\tilde{u} \in t^\lambda \mathbb{C}[\log t]\mathcal{A}^m_M$

of $\mathcal{M}$ with the same property, the leading exponent λ' of $u - \tilde{u}$ satisfies $\lambda' - e(\bar{u}_i) \in \mathbb{N}^n \setminus \{0\}$, which contradicts to (20). Thus we have proved the required uniqueness of the solution.

Next suppose $u_i = t^\lambda v_i(\log t)$ with $\deg v_i > 0$. Let V be a vector space spanned by the components of elements of $\mathbb{C}[\partial_\xi]v_i(\xi)$ and let $\{w_1(\xi), \dots, w_q(\xi)\}$ be a basis of V. Here we may assume $\mathbb{C}[\partial_\xi]w_k \in \sum_{\nu=1}^k \mathbb{C}w_\nu$ for $k = 1, \dots, q$. Let $\hat{u}$ be the vector of size qm with components $\hat{u}_\nu w_\nu(\log t)$ with $\hat{u}_\nu \in t^\lambda \mathcal{A}_M^m$ for $\nu = 1, \dots, q$. Then the system $\mathcal{M}$ is replaced by a system $\hat{\mathcal{M}}$ with an unknown function $\hat{u}$ where P_i are replaced by suitable $\hat{P}_i \in M(qm, \mathcal{D}_*)$, respectively. We note that $\hat{\mathcal{M}}$ also satisfies the assumption of the theorem because $\det\bigl(\sigma_*(\hat{P}_i)\bigr) = \det\bigl(\sigma_*(P_i)\bigr)^q$. Thus we may only consider the solutions with components in $t^\lambda \mathcal{A}_M$.

For example, if $n = n' = 1$ and $P = (\vartheta - \lambda)^2 + t^2\partial_x^2$, the solution of the equation $Pu = 0$ in the space $t^\lambda \mathcal{A}_M \oplus (t^\lambda \log t)\mathcal{A}_M$ corresponds to the solution of

$$\begin{pmatrix} (\vartheta - \lambda)^2 + t^2\partial_x^2 & 2(\vartheta - \lambda) \\ & (\vartheta - \lambda)^2 + t^2\partial_x^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

in the space $t^\lambda \mathcal{A}_M^2$ by the correspondence $u = u_1 + u_2 \log t$.

To complete the proof of the system we have only to prove that the map (21) is surjective. Let u be any ideally analytic solution of $\mathcal{M}$. Then any leading exponent of u is an exponent of the system $\mathcal{M}$ and therefore we define $\phi_i(x)$ by (19) if $e(\bar{u}_i)$ is a leading exponent of u and by 0 otherwise. Then if $u \neq \sum_i T_{\bar{u}_i}(\psi)$, any leading exponent of $u - \sum_i T_{\bar{u}_i}(\phi)$ is not in the set $\{e(\bar{u}_i)\}$, which contradicts the first claim in the theorem. □

We will return to the case when (16) is the complete system which has a holomorphic parameter $z \in U \subset \mathbb{C}^\ell$.

First assume that $\overline{\mathcal{M}}$ is semisimple for any $z \in U$ (cf. Definition 5.2) and that the indicial equation $\overline{\mathcal{M}}$ satisfies the assumption in Proposition 5.3 by putting $t_i = \mathrm{e}^{-x_i}$ for $1 \le i \le n$. Then the proof of the previous theorem implies the following.

Proposition 6.4. *Assume that $\overline{\mathcal{M}}$ is semisimple for any $z \in U$. Let $\{\bar{u}_i(x, z) = t^{\lambda_i(z)} f_i(z)\,;\, i = 1, \dots, r\}$ be a basis of the solutions of* (17) *for any $z \in U$. Here $f_i(z) \in \mathcal{O}(U)^m$. Assume* (20) *for any $z \in U$. Then $T_{\bar{u}_i}(\phi)$ is holomorphic for $z \in U$ under the notation in Theorem 6.3.*

To examine the case without the assumption in this proposition, we study a generic holomorphic curve $t \mapsto z(t)$ through the point $z^o \in U$ where the assumption breaks. Hence we restrict the case when $\ell = 1$.

Suppose $\ell = 1$ and fix $z^o \in U$. For simplicity we put $z^o = 0$. Assume that $\mathcal{M}$ is semisimple (cf. Definition 5.2) for any fixed $z \in U \setminus \{0\}$. We will shrink U if necessary hereafter until the end of the following theorem. Let $\{\bar{u}_1, \dots, \bar{u}_r\}$ be a basis of the solutions of the indicial equation for $\forall z \in U \setminus \{0\}$, where $\bar{u}_i$ are

$$\bar{u}_i(t, z) = t^{\lambda_i(z)} f_i(z) \quad \text{for} \quad i = 1, \dots, r$$

with suitable $f_i \in \mathcal{O}(U)^m$. Then Proposition 5.3 assures that there exist meromorphic functions $c_{ij}(z)$ such that by denoting

$$\bar{w}_i(t,z) = \sum_{j=1}^{r} c_{ij}(z)\bar{u}_j(t,z),$$

$\{\bar{w}_1,\ldots,\bar{w}_r\}$ is a basis of the solutions of the indicial equation for $\forall z \in U$ and $\bar{w}_j(t,z)$ are holomorphic function of $(\log t, z) \in \mathbb{C}^n \times U$. By virtue of (15), we may assume $c_{ij}(z) = 0$ if $\lambda_i(0) \neq \lambda_j(0)$.

Then we have the following theorem which is the main purpose of this note.

Theorem 6.5. *Under the notation above. there exist differential operators $R_{ij}(x,z,\partial_x)$ such that for any $\phi(x,z) \in {}_U\mathcal{A}_M^m$, $\sum_{i=1}^r T_{\bar{u}_i}\bigl(R_{ij}(x,z,\partial_x)\phi(x,z)\bigr)$ is a holomorphic function of $z \in U$ and an ideally analytic solution of $\mathcal{M}$ with the complete leading term $\phi(x)\bar{w}_i(t,z)$ for any fixed $z \in U$. Moreover the map*

$$\begin{aligned} \mathcal{A}_N^r &\xrightarrow{\sim} \{\textit{ideally analytic solutions of } \mathcal{M}\},\\ \bigl(\phi_i(x)\bigr) &\mapsto \sum_{i,j} T_{\bar{u}_i}\bigl(R_{ij}(x,z,\partial_x)\phi_j(x)\bigr) \end{aligned}$$

holomorphically depends on $z \in U$ and it is bijective for any $z \in U$. Here $R_{ij}(x,z,\partial_x)$ are holomorphic functions of $z \in U \setminus \{0\}$ valued in the space of differential operators on N and may have at most poles at $z = 0$ and moreover

$$R_{ij}(x,z,\partial_x) = \begin{cases} 0 & \textit{if } \ \lambda_i(0) - \lambda_j(0) \notin \mathbb{N}^n, \\ c_{ij}(z) & \textit{if } \ \lambda_i(0) = \lambda_j(0). \end{cases}$$

Proof. We will inductively construct $R_{ij}(x,z,\partial_x)$ according to the number $L(\lambda_j) = \sum_{\nu=1}^n \Re\lambda_{j,\nu}(0)$. Here $\lambda_j = (\lambda_{j,1},\ldots,\lambda_{j,n})$ and $\Re\zeta$ denotes the real part of $\zeta \in \mathbb{C}$.

Fix a positive integer k with $k \le r$. By the hypothesis of the induction we may assume that R_{ij} have been constructed if $L(\lambda_j) > L(\lambda_k)$. Put $R_{jk}^{(0)} = c_{ik}(z)$. We inductively define $R_{ik}^{(\nu)}$ for $\nu = 0,1,\ldots$ as follows. Put

$$\sum_{i=1}^r T_{\bar{u}_i}\bigl(R_{ik}^{(\nu)}\phi(x,z)\bigr) = z^{-n_\nu}\phi_{n_\nu}^{(\nu)}(t,x) + \cdots + z^{-1}\phi_1^{(\nu)}(t,x) + \phi_0^{(\nu)}(t,x,z)$$

with $\phi_0^{(\nu)}(t,x,z) \in {}_U\mathcal{A}_M$. Suppose $n_\nu > 0$. By the analytic continuation of $z^{n_\nu}\sum_i T_{\bar{u}_i}\bigl(R_{ik}^{(\nu)}\phi(x,z)\bigr)$, it is clear that $\phi_{n_\nu}^{(\nu)}(t,x)$ is a solution of $\mathcal{M}$ at $z = 0$. Any leading exponent μ of $\phi_{n_\nu}^{(\nu)}(t,x)$ satisfies $\mu - \lambda_k(0) \in \mathbb{N}^n \setminus \{0\}$ and hence the complete leading term of $\phi_{n_\nu}^{(\nu)}(t,x)$ is

$$\sum_{\lambda_j(0) \in \lambda_k(0) + \left(\mathbb{N}^n \setminus \{0\}\right)} \psi_j^{(\nu)}(x)\bar{w}_j(t,0).$$

Note that $\psi_j^{(\nu)}(x) = P_j^{(\nu)}(x, \partial_x)\phi(x)$ for some differential operators which do not depend on $\phi(x)$. Put $P_j^{(\nu)}(x, \partial_x) = 0$ if $\lambda_j(0) - \lambda_k(0) \notin \mathbb{N}^n \setminus \{0\}$. Hence

$$\sum_{i=1}^{r} T_{\bar{u}_i}\bigl(R_{ik}^{(\nu)}(x, z, \partial_x)\phi(x)\bigr) - \sum_{i=1}^{r}\sum_{j=1}^{r} z^{-n_\nu} T_{\bar{u}_i}\bigl(R_{ij}(x, z, \partial_x)P_j^{(\nu)}(x, \partial_x)\phi(x)\bigr) \tag{22}$$

has a pole of order less than n_k. Defining

$$R_{ik}^{(\nu+1)}(x, z, \partial_x) = R_{ik}^{(\nu)}(x, z, \partial_x) - \sum_{j=0}^{r} z^{-n_\nu} R_{ij}(x, z, \partial_x)P_j^{(\nu)}(x, \partial_x)$$

inductively, we have $R_{ij}(x, z, \partial_x) = R_{ij}^{(\nu)}(x, z, \partial_x)$ for certain ν such that the left hand size of (22) is holomorphic at $z = 0$. $\square$

Remark 6.6. Let $P_i \in \mathcal{D}_*$ for $i = 1, \ldots, n$ satisfies

$$\begin{cases} [P_i, P_j] = \sum_{\nu=1}^{n} R_{ij\nu}P_\nu \quad \text{for } \ 1 \le i \le j \le n, \\ \sigma_*(P_i) \text{ do not depend on } x \in N, \\ \{\xi \in \mathbb{C}^n\,;\, \bar{\sigma}_*(P_1)(\xi) = \cdots = \bar{\sigma}_*(P_n)(\xi) = 0\} = \{0\} \end{cases}$$

with some $R_{ij\nu} \in \mathcal{D}_*$ satisfying $\sigma_*(R_{ij\nu}) = 0$. Then for a suitable positive integer L there exist $R_i \in \mathbb{C}[\vartheta]$ such that

$$\begin{cases} \operatorname{ord} P_i + \operatorname{ord} R_i = 2L, \\ \sigma_*(P_0) = \xi_1^{2L} + \cdots + \xi_n^{2L} \end{cases}$$

by putting

$$P_0 = \sum_{i=1}^{n} R_i P_i.$$

Then $\{P_0, \ldots, P_n\}$ satisfies (11) with $S = 0$ and $\sigma_*(T_{ij}) = 0$ because

$$[P_0, P_j] = \sum_{i=1}^{n}\Bigl([P_0, R_j]P_i + \sum_{\nu=1}^{n} R_i R_{ij\nu} P_\nu\Bigr)$$

and $\sigma_*([P_0, R_i]) = \sigma_*(R_i R_{ij\nu}) = 0$.

In this case let λ^o be an exponent of the system $P_i u = 0$ $(1 \le i \le n)$. Then for a suitable $\rho \in \mathbb{C}^n$ and a positive integer k, the system

$${}_U\mathcal{M}:\ \bigl(P_i - \sigma_*(P)(\lambda^o + \rho z^k)\bigr)u = \sum R_i\bigl(P_i - \sigma_*(P)(\lambda^o + \rho z^k)\bigr)u = 0$$

satisfies the assumption of Theorem 6.5 for $U = \{z \in \mathbb{C}\,;\, |z| < 1\}$ by changing the lower order terms of R_i if necessary. Hence we can analyze the ideally analytic solutions of $\mathcal{M}$ by the analytic continuation of the parameter z to the origin.

Theorem 6.7. *Retain the notation and the assumption in Theorem 6.5. Let r' be the dimension of the finitely generated $\mathbb{C}[\vartheta]$-module*

$$\bar{\mathcal{M}}^o := \sum_{j=1}^{m} \mathbb{C}[\vartheta]u_j \Big/ \sum_{i=0}^{q}\sum_{k=1}^{m} \mathbb{C}[\vartheta] \sum_{j=1}^{m} \bar{\sigma}_*(P_i)_{kj}(z^o,\vartheta)u_j.$$

Suppose $n' = 0$ and $r' \le r$. Then $r' = r$ and any solution of $\mathcal{M}$ defined on a small connected neighborhood of $(t^o, x^o) \in M$ with $z = z^o$ and $0 < |t^o_j| \ll 1$ for $j = 1, \dots, n$ is an ideally analytic solution given in Theorem 6.5. In particular the dimension of space of the solutions equals r.

Proof. *Let w_ν for $\nu = 1, \dots, r'$ be elements of $\sum_{j=1}^m \mathbb{C}[\vartheta]u_j$ whose residue classes form a basis of $\bar{\mathcal{M}}^o$. Fix $z = z^o$. Then in a neighborhood of $(0, x^o)$*

$$\sum_{j=1}^{r} \mathcal{A}_M[\vartheta]u_j = \sum_{\nu=1}^{r'} \mathcal{A}_M w_\nu + \sum_{i=0}^{q}\sum_{k=1}^{m} \mathcal{A}_M[\vartheta]\sum_{j=1}^{m}(P_i)_{kj}u_j.$$

Let w be a column vector of size r' with components w_ν. Then the system $\mathcal{M}$ implies

$$\mathcal{N} \,:\, \vartheta_j w = Q_j(t)w \quad \text{for} \;\; j = 1, \dots, n$$

with suitable $Q_j \in M(r', \mathcal{A}_M)$. Then any solution $w(t)$ of $\mathcal{N}$ on a neighborhood of (t^o, x^o) is analytic and $w = 0$ if $w(t^o) = 0$. Hence the dimension of the space of solutions of $\mathcal{N}$ is smaller than or equals to r'. But we have constructed r linearly independent solutions in Theorem 6.5. Hence we have this theorem. □

Remark 6.8. Retain the notation in Theorem 6.7. Suppose $q = n-1$, $[P_i, P_j] = 0$ for $0 \le i < j \le q$, $\bar{\sigma}_*(P_i)$ are diagonal matrices and

$$\{\xi \in \mathbb{C}^n \,;\, \bar{\sigma}_*(P_i)(\xi) = 0 \quad \text{for} \;\; i = 0, \dots, q\} = \{0\}.$$

Then $r' = r$ and $r' = m\prod_{i=0}^{q} \operatorname{ord} P_i$.

7 Examples related to $SL(n, \mathbb{R})$

For a connected real reductive Lie group G and an open subgroup H of the fixed point group of an involutive automorphism σ of G, the homogeneous space G/H is called a *reductive symmetric homogeneous space*. Then in a suitable realization $\widetilde{X}$ of G/H constructed by [O6], the system of differential equations that defines the simultaneous eigenspace of the elements of the ring $\mathbb{D}(G/H)$ of the invariant differential operators on G/H has regular singularities along the boundaries of G/H in this realization. It is an important problem to study the eigenspace. For example, see [K–] in the cases of Riemannian symmetric spaces.

Note that the Lie group G is identified with a symmetric homogeneous space of $G\times G$ with respect to the involutive automorphism σ of G defined by $\sigma(g_1,g_2)=(g_2,g_1)$ for $(g_1,g_2)\in G_1\times G_2$ and that any irreducible admissible representation of G can be realized in an eigenspace of $\mathbb{D}(G)$.

In this section we will consider differential equations related to the Lie group $G=SL(n,\mathbb{R})$, which give examples of the differential equations we study in this note. The element of the Lie algebra $\mathfrak{sl}(n,\mathbb{R})$ of G is identified with that of $M(n,\mathbb{R})$ whose trace equals 0. Let E_{ij} be the fundamental matrix unit whose (i,j)-component equals 1 and the other components are 0. Then $\mathfrak{sl}(n,\mathbb{R})$ is spanned by the elements $\tilde E_{ij}=E_{ij}-\frac{\delta_{ij}}{n}(E_{11}+\cdots+E_{nn})$ with $1\le i\le j\le n$. For simplicity we put $\tilde E_i=\tilde E_{ii}$.

We identify $\mathfrak{sl}(n,\mathbb{R})$ with the space of right invariant vector field on G by

$$(Xf)(g)=\frac{\mathrm{d}}{\mathrm{d}t}f(g\mathrm{e}^{tX})\Big|_{t=0}\quad\text{for } X\in\mathfrak{sl}(n,\mathbb{R}),\ f\in C^\infty(G)\text{ and } g\in G.$$

Here we note that

$$(E_{pq}f)\bigl((x_{ij})\bigr):=\frac{\mathrm{d}}{\mathrm{d}t}f\bigl((x_{ij})\mathrm{e}^{tE_{pq}}\bigr)\Big|_{t=0}=\Bigl(\sum_{\nu=1}^n x_{\nu p}\frac{\partial f}{\partial x_{\nu q}}\Bigr)\bigl((x_{ij})\bigr)$$

for $g\in C^\infty\bigl(GL(n,\mathbb{R})\bigr)$ and $\bigl(x_{ij}\bigr)_{\substack{1\le i\le n\\1\le j\le n}}\in GL(n,\mathbb{R})$ because (i,j)-component of $\bigl(x_{ij}\bigr)_{\substack{1\le i\le n\\1\le j\le n}}E_{pq}$ equals $x_{ip}\delta_{qj}$.

We first review, by examples, that the invariant differential operators of the Riemannian symmetric space G/K has regular singularities along the boundaries of the space in the realization constructed in [O2]. By the Iwasawa decomposition $G=\bar NAK$ with

$$\begin{aligned}
&K=SO(n)=\{g\in SL(n,\mathbb{R})\,;\,{}^tgg=I_n\},\\
&A=\left\{a=\begin{pmatrix}a_1&&\\&\ddots&\\&&a_n\end{pmatrix}\,;\,a_j>0\ \text{ for } 1\le j\le n\text{ and } a_1\cdots a_n=1\right\},\qquad(23)\\
&\bar N=\left\{\begin{pmatrix}1&&&\\x_{21}&1&&\\\vdots&\vdots&\ddots&\\x_{n1}&x_{n2}&\cdots&1\end{pmatrix}\,;\,x_{ij}\in\mathbb{R}\ \text{ for } 1\le j<i\le n\right\},\\
&t_j:=\frac{a_{j+1}}{a_j}\qquad\text{for } j=1,\dots,n-1,
\end{aligned}$$

the Riemannian symmetric space G/K is identified with the product manifold $\bar N\times A$ with the coordinate $(t_k,x_{ij})\in(0,\infty)^{n-1}\times\mathbb{R}^{\frac{n(n-1)}{2}}$. Then the Lie algebra of the solvable group of $\bar NA$ is spanned by the elements

$$E_{ij}=\Bigl(\prod_{\nu=j}^{i-1}t_\nu\Bigr)\Bigl(\frac{\partial}{\partial x_{ij}}+\sum_{\nu=i+1}^{n}x_{\nu i}\frac{\partial}{\partial x_{\nu j}}\Bigr)\quad\text{for } 1\le j<i\le n,$$
$$\tilde E_{ij}=E_{ij}-\frac{\delta_{ij}}{n}(E_{11}+\cdots+E_{nn})\quad\text{for } 1\le i\le n \text{ and } 1\le j\le n,$$
$$E_i:=\tilde E_{ii}=\vartheta_{i-1}-\vartheta_i\quad\text{for } 1\le i\le n,\quad \vartheta_0=\vartheta_{n+1}=0.$$

The coordinate $(t_k,x_{ij})\in\mathbb{R}^{\frac{(n+2)(n-1)}{2}}$ can be used for local coordinate of the realization of G/K.

Let $U(\mathfrak{g})$ be the universal enveloping algebra of the complexification $\mathfrak{g}$ of the Lie algebra of G. Then if $G=SL(n,\mathbb{R})$, the ring $\mathbb{D}(G/K)$ is naturally isomorphic to the center $U(\mathfrak{g})^G$ of $U(\mathfrak{g})$ and $U(\mathfrak{g})^G$ is generated by the elements $L_2,\ldots,L_n$ which are given by

$$\det\bigl(\tilde E_{ij}+(\tfrac{n+1}{2}-i-\lambda)\delta_{ij}\bigr)=L_n-L_{n-1}\lambda+\cdots+(-1)^n\lambda^n$$

for $\lambda\in\mathbb{C}$ (cf. [Ca]). Here $\det(A_{ij})=\sum_{\sigma\in\mathfrak{S}_n}\operatorname{sgn}(\sigma)A_{\sigma(1)1}\cdots A_{\sigma(n)n}$ and $U(\mathfrak{g})^G$ is generated by the algebraically independent $(n-1)$-elements which are the coefficients of λ^k for $k=0,1,\ldots,n-2$.

Let $\mathfrak{k}$ be a Lie algebra of $SO(n)$, which is generated by the elements $E_{ij}-E_{ji}$ for $1\le i<j\le n$.

Since

$$\begin{aligned}\Delta_2&=\det\begin{pmatrix}E_1+\frac12 & E_{12}\\ E_{21} & E_2-\frac12\end{pmatrix}=(E_1+\tfrac12)(E_2-\tfrac12)-E_{21}E_{12}\\ &\equiv(E_1+\tfrac12)(E_2-\tfrac12)-E_{21}^2\mod U(\mathfrak{g})\mathfrak{k}\\ &=-(\vartheta-\tfrac12)^2-t^2\partial_x^2=-t^2(\partial_t^2+\partial_x^2)-\tfrac14\quad\text{with}\ \ \vartheta=t\tfrac{\partial}{\partial t},\end{aligned}$$

we see that $\mathbb{D}\bigl(SL(2,\mathbb{R})/SO(2)\bigr)=\mathbb{C}[t^2\bigl(\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}\bigr)]$. Here $SL(2,\mathbb{R})/SO(2)$ is realized in the upper half plane $\{x+\mathrm{i}t\,;\,(t,x)\in(0,\infty)\times\mathbb{R}\}$ and Δ_2 has regular singularities along the real axis. On the other hand, the explicit form of the vector field L_X defined by the translation $e^{-sX}\cdot p$ for $s\in\mathbb{R}$, $X\in\mathfrak{g}$ and $p\in SL(2,\mathbb{R})/SO(2)$ is given by

$$L_{E_{21}}=-\partial_x,\ L_{E_1}=\vartheta+x\partial_x,\ L_{E_{12}}=2x\vartheta-(t^2-x^2)\partial_x.$$

When $G=SL(3,\mathbb{R})$, we have

$$\begin{aligned}&\det\begin{pmatrix}E_1+1-\lambda & E_{12} & E_{13}\\ E_{21} & E_2-\lambda & E_{23}\\ E_{31} & E_{32} & E_3-1-\lambda\end{pmatrix}=(E_1+1-\lambda)(E_2-\lambda)(E_3-1-\lambda)\\ &\quad+E_{21}E_{32}E_{13}+E_{31}E_{12}E_{23}-(E_{11}+1-\lambda)E_{32}E_{23}-E_{21}E_{12}(E_3-1-\lambda)\\ &\quad-E_{31}(E_2-\lambda)E_{13}=\Delta_3-\Delta_2\lambda-\lambda^3\end{aligned}$$

with

$$\begin{aligned}
\Delta_3 &= (E_1+1)E_2(E_3-1) + E_{21}E_{32}E_{13} + E_{31}E_{12}E_{23}\\
&\quad - (E_1+1)E_{32}E_{23} - E_{21}E_{12}(E_3-1) - E_{31}E_2E_{13}\\
&\equiv (E_1+1)E_2(E_3-1) - (E_1+1)E_{32}^2 - (E_3-1)E_{21}^2 - (E_2-1)E_{31}^2\\
&\qquad + 2E_{21}E_{32}E_{31} \mod U(\mathfrak{g})\mathfrak{k}\\
&= -(\vartheta_1-1)(\vartheta_1-\vartheta_2)(\vartheta_2-1) + 2t_1^2t_2^2(\partial_x + y\partial_z)\partial_y\partial_z\\
&\qquad + (\vartheta_1-1)t_2^2\partial_y^2 - (\vartheta_1-\vartheta_2-1)t_1^2t_2^2\partial_z^2 - (\vartheta_2-1)t_1^2(\partial_x+y\partial_z)^2,\\
\Delta_2 &= E_2(E_3-1) + (E_1+1)(E_3-1) + (E_1+1)E_2\\
&\quad - E_{32}E_{23} - E_{21}E_{12} - E_{31}E_{13}\\
&\equiv E_2(E_3-1) + (E_1+1)(E_3-1) + (E_1+1)E_2\\
&\quad - E_{32}^2 - E_{21}^2 - E_{31}^2 \mod U(\mathfrak{g})\mathfrak{k}\\
&= -(\vartheta_1-1)^2 + (\vartheta_1-1)(\vartheta_2-1) - (\vartheta_2-1)^2\\
&\quad - t_2^2\partial_y^2 - t_1^2t_2^2\partial_z^2 - t_1^2(\partial_x+y\partial_z)^2,\\
x &= x_{21},\ y = x_{32} \text{ and } z = x_{31}.
\end{aligned}$$

Then $\mathbb{D}\big(SL(3,\mathbb{R})/SO(3)\big) = \mathbb{C}[\bar{\Delta}_3, \bar{\Delta}_2]$, where $\bar{\Delta}_3$ and $\bar{\Delta}_2$ are the last expressions of Δ_3 and Δ_2 in the above, respectively. This expression of invariant differential operators on $SL(3,\mathbb{R})/SO(3)$ is given by [O1] to obtain the Poisson integral representation of any simultaneous eigenfunction of the operators on the space, where such representation is first obtained in the space with the rank larger than one. In fact $4\Delta_2$ and $8\Delta_2 + 8\Delta_3$ are explicitly written there under the coordinate (s,t,u,v,w) with $(s,t,u,v,w) = (t_2^2, t_1^2, x, y, z)$, which corresponds to a local coordinate system in the realization given in [OS].

When $G = SL(n,\mathbb{R})$ the second order element L_2 of $U(\mathfrak{g})^G$ is

$$\begin{aligned}
L_2 &= \sum_{1\le i<j\le n} \Big((E_i + \frac{n+1}{2} - i)(E_j + \frac{n+1}{2} - j) - E_{ji}E_{ij}\Big)\\
&\equiv \sum_{1\le i<j\le n} (\tilde{\vartheta}_{i-1} - \tilde{\vartheta}_i)(\tilde{\vartheta}_{j-1} - \tilde{\vartheta}_j)\\
&\quad - \sum_{1\le i<j\le n} \Big(\prod_{\nu=i}^{j-1} t_\nu^2\Big)\Big(\frac{\partial}{\partial x_{ji}} + \sum_{\nu=j+1}^{n} x_{\nu j}\frac{\partial}{\partial x_{\nu i}}\Big)^2 \mod U(\mathfrak{g})\mathfrak{k},\\
\tilde{\vartheta}_i &= \vartheta_i - \tfrac{i(n-i)}{2}
\end{aligned}$$

and $\mathbb{D}(G/K) = \mathbb{C}[\bar{L}_2, \ldots, \bar{L}_n]$ satisfying

$$\begin{aligned}
\sigma_*(\bar{L}_k) &= \sum_{1\le i_1<i_2<\cdots<i_k\le n} (\tilde{\xi}_{i_1-1} - \tilde{\xi}_{i_1})(\tilde{\xi}_{i_2-1} - \tilde{\xi}_{i_2})\cdots(\tilde{\xi}_{i_k-1} - \tilde{\xi}_{i_k}),\\
\tilde{\xi}_i &= \xi_i - \tfrac{i(n-i)}{2}
\end{aligned}$$

for $k = 2, \ldots, n$.

We will examine more examples. For a in (23) we have

$$\operatorname{Ad}(a^{-1})E_{ij} := aE_{ij}a^{-1} = a_i^{-1}a_jE_{ij} = t_{ij}E_{ij},$$
$$t_{ij} = a_i^{-1}a_j = \begin{cases} t_it_{i+1}\cdots t_{j-1} & \text{if } i \le j,\\ t_j^{-1}t_{j+1}^{-1}\cdots t_{i-1}^{-1} & \text{if } i > j,\end{cases}$$
$$U(\mathfrak{g})\mathfrak{k} = \sum_{1\le i<j\le n} U(\mathfrak{g})(E_{ij}-E_{ji}).$$

Hence

$$\begin{aligned}
&\operatorname{Ad}(a^{-1})(E_{ij}-E_{ji})^2 + (E_{ij}-E_{ji})^2\\
&\qquad - (t_{ij}+t_{ij}^{-1})\operatorname{Ad}(a^{-1})(E_{ij}-E_{ji})\cdot(E_{ij}-E_{ji})\\
&\quad = (t_{ij}^2-1)E_{ij}E_{ji} + (t_{ij}^{-2}-1)E_{ji}E_{ij}\\
&\quad = (t_{ij}-t_{ij}^{-1})^2E_{ji}E_{ij} + (t_{ij}^2-1)(E_{ii}-E_{jj}),\\
&\operatorname{Ad}(a^{-1})(E_{ij}-E_{ji})\cdot E_{ij} - t_{ij}E_{ij}^2 = -t_{ij}^{-1}E_{ji}E_{ij}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
E_{ji}E_{ij} &= \frac{t_{ij}^2}{1-t_{ij}^2}(E_{ii}-E_{jj}) \qquad (24)\\
&\quad + \frac{t_{ij}^2}{(1-t_{ij}^2)^2}\big(\operatorname{Ad}(a^{-1})(E_{ij}-E_{ji})^2 + (E_{ij}-E_{ji})^2\big)\\
&\quad - \frac{t_{ij}(1+t_{ij}^2)}{(1-t_{ij}^2)^2}\operatorname{Ad}(a^{-1})(E_{ij}-E_{ji})\cdot(E_{ij}-E_{ji})\\
&= t_{ij}^2E_{ij}^2 - t_{ij}\operatorname{Ad}(a^{-1})(E_{ij}-E_{ji})\cdot E_{ij}. \qquad (25)
\end{aligned}$$

Let (ϖ, V_ϖ) be a finite dimensional representation of a closed subgroup H of G and $C^\infty(G;V_\varpi)$ denote the space of V_ϖ-valued C^∞-functions on G. Then the space of C^∞-sections $C^\infty(G/H;\varpi)$ of the G-homogeneous bundle associated to ϖ is

$$\{f \in C^\infty(G;V_\varpi)\,;\, f(gh) = \varpi^{-1}(h)f(g) \text{ for } \forall h\in H\}.$$

Consider the case when $H = K$. Because of the decomposition $G = KAK$ the function $f \in C^\infty(G/K;\varpi)$ is determined by its restriction on KA and by the natural map $K\times A \to KA$ the restriction can be considered as a function $\bar f$ on $K\times A$. Then the action of the differential operator L_2 to $\bar f$ is

$$\begin{aligned}
\bar L_2 = \sum_{1\le i<j\le n}\Big(&(\tilde\vartheta_{i-1}-\tilde\vartheta_i)(\tilde\vartheta_{j-1}-\tilde\vartheta_j) - \frac{t_{ij}^2}{1-t_{ij}^2}(\vartheta_{i-1}-\vartheta_i-\vartheta_{j-1}+\vartheta_j)\\
&- \frac{t_{ij}^2}{(1-t_{ij}^2)^2}\big(\operatorname{Ad}(a^{-1})(E_{ij}-E_{ji})^2 + \varpi(E_{ij}-E_{ji})^2\big)\\
&- \frac{t_{ij}(1+t_{ij}^2)}{(1-t_{ij}^2)^2}\operatorname{Ad}(a^{-1})(E_{ij}-E_{ji})\cdot\varpi(E_{ij}-E_{ji})\Big)
\end{aligned}$$

at $(k,a)\in K\times A$, which follows from (24). Here the induced representation of the Lie algebra $\mathfrak{k}$ of K is also denoted by ϖ.

Let (δ, V_δ) be an irreducible representation of K. Then the δ-component of $C^\infty(G/K;\varpi)$ is an element $f\in V\otimes C^\infty(G/K;\varpi)$ which satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}f(\mathrm{e}^{tX}g)\Big|_{t=0}=\big(\delta(X)f\big)(g)$$

for $X\in\mathfrak{k}$. Hence the function f is determined by its restriction $\bar f$ on A and the action of the operator L_2 to $\bar f$ is

$$\begin{aligned}\bar L_2=\sum_{1\le i<j\le n}\Big(&(\tilde\vartheta_{i-1}-\tilde\vartheta_i)(\tilde\vartheta_{j-1}-\tilde\vartheta_j)-\frac{t_{ij}^2}{1-t_{ij}^2}(\vartheta_{i-1}-\vartheta_i-\vartheta_{j-1}+\vartheta_j)\\&-\frac{t_{ij}^2}{(1-t_{ij}^2)^2}\big(\delta(E_{ij}-E_{ji})^2+\varpi(E_{ij}-E_{ji})^2\big)\\&-\frac{t_{ij}(1+t_{ij}^2)}{(1-t_{ij}^2)^2}\delta(E_{ij}-E_{ji})\otimes\varpi(E_{ij}-E_{ji})\Big).\end{aligned}\tag{26}$$

Note that the operator $P=\bar L_2$ satisfies the assumption of Corollary 2.3.

When G is $SL(2,\mathbb{R})$ or its universal covering group and $\bar f$ is an eigenfunction of L_2, we can put $\varpi(E_{12}-E_{21})=\sqrt{-1}k$ and $\delta(E_{12}-E_{21})=-\sqrt{-1}m$ for certain numbers k and m and

$$\Big(\vartheta^2+\frac14-\frac{1+t^2}{1-t^2}\vartheta+\frac{t(k-mt)(m-kt)}{(1-t^2)^2}-(\lambda+\frac12)^2\Big)\bar f=0.$$

Put $t=\mathrm{e}^{-x}$ and $u=\bar f$. Then $\vartheta=-\frac{\mathrm{d}}{\mathrm{d}x}$ and

$$\begin{aligned}&u''+\coth x\cdot u'-\frac{(k+m)^2}{4\sinh^2x}u+\frac{km}{4\sinh^2\frac x2}u=\lambda(\lambda+1)u,\\&\frac{\mathrm{d}^2v}{\mathrm{d}z^2}-\frac{(k+m-1)(k+m+1)}{\sinh^2 2z}v+\frac{km}{\sinh^2 z}v=\big(2\lambda+1\big)^2v.\end{aligned}$$

by denoting $v=\sinh^{\frac12}x\cdot u$ and $z=\frac x2$.

Then for $\tilde v=\sinh^m z\cdot\sinh^{-\frac{k+m+1}{2}}2z\cdot v$ and $w=-\sinh^2 z$ we have

$$w(1-w)\frac{\mathrm{d}^2\tilde v}{\mathrm{d}w^2}+\Big(\frac{k-m+2}{2}-(k+2)w\Big)\frac{\mathrm{d}\tilde v}{\mathrm{d}w}-\Big(\frac k2-\lambda\Big)\Big(\frac k2+\lambda+1\Big)\tilde v=0$$

and hence $\bar f$ is a linear combination of the functions

$$\begin{cases}\sinh^{\frac{k-m}{2}}z\cdot\cosh^{\frac{k+m}{2}}z\cdot F(\frac k2-\lambda,\frac k2+\lambda+1,\frac{k-m}{2}+1;-\sinh^2 z),\\ \sinh^{\frac{m-k}{2}}z\cdot\cosh^{\frac{k+m}{2}}z\cdot F(\frac m2-\lambda,\frac m2+\lambda+1,\frac{m-k}{2}+1;-\sinh^2 z\big).\end{cases}$$

Thus it is clear that the non-zero real analytic solution $\bar f$ defined in a neighborhood of the point $z=0$ exists if and only if $k-m\in 2\mathbb{Z}$. Here $F(\alpha,\beta,\gamma;z)$ denotes the Gauss hypergeometric function (cf. [W]).

Next we assume $H = N$ and ϖ is a character of N. Then there exist complex numbers $c_1, \ldots, c_{n-1}$ such that

$$\varpi(\mathrm{e}^{\sum_{1\le i<j\le n} s_{ij}E_{ij}}) = \mathrm{e}^{\sqrt{-1}(c_1 s_{12}+\cdots+c_{n-1}s_{n-1,n})}.$$

The element $f \in C^\infty(G/N;\varpi)$ is determined by the restriction $\bar f = f|_{KA}$ and it follows from (25) that the operation of L_2 to $\bar f$ is

$$\sum_{1\le i<j\le n} (\tilde\vartheta_{i-1}-\tilde\vartheta_i)(\tilde\vartheta_{j-1}-\tilde\vartheta_j) + \sum_{1\le i<n} \bigl(c_i^2 t_i^2 + \sqrt{-1}c_i t_i(E_{i,i+1}-E_{i+1,i})\bigr). \quad (27)$$

Hence if $G = SL(2,\mathbb{R})$, the eigenfunction f of L_2 of the δ-component of $C^\infty(G/N;\varpi)$ with $\delta(E_{12}-E_{21}) = \sqrt{-1}m$ satisfies

$$\Bigl(-(\vartheta-\tfrac12)^2 + c_1^2t^2 - c_1mt + (\lambda+\tfrac12)^2\Bigr)f|_A = 0$$

and hence

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(f|_A) - \Bigl(c_1^2 - \frac{c_1m}{t} + \frac{\lambda(\lambda+1)}{t^2}\Bigr)(f|_A) = 0.$$

If we put $u(x) = \mathrm{e}^{\frac{x}{2}}\bigl(f|_A(\mathrm{e}^{-x})\bigr)$, then

$$u'' - \bigl(c_1^2\mathrm{e}^{-2x} - c_1m\mathrm{e}^{-x}\bigr)u = (\lambda+\tfrac12)^2u.$$

Denoting $W(\pm 2c_1t) = f|_A(t)$, we have the Whittaker equation (cf. [W])

$$W'' + \Bigl(-\frac14 \pm \frac{m}{2t} + \frac{\frac14-(\lambda+\frac12)^2}{t^2}\Bigr)W = 0.$$

8 Completely integrable quantum systems

A Schrödinger operator

$$P = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} + R(x_1,\ldots,x_n)$$

of n variables is called *completely integrable* if there exist n algebraically independent differential operators P_k such that

$$[P_i,P_j] = 0 \quad \text{for} \quad 1\le i<j\le n \quad \text{and} \quad P\in\mathbb{C}[P_1,\ldots,P_n].$$

Under the coordinate system $(t_1,\ldots,t_n)$ with

$$t_1 = \mathrm{e}^{x_1-x_2},\ldots,t_{n-1} = \mathrm{e}^{x_{n-1}-x_n}, t_n = \mathrm{e}^{x_n},$$

the Schrödinger operators P which belong to $\mathcal{D}_*$ and have elements $Q\in\mathcal{D}_*$ satisfying

$$Q = \sum_{k=1}^{n} \frac{\partial^4}{\partial x_k^4} + Q' \quad \text{with} \quad \operatorname{ord} Q' < 4$$

are classified in [O8] and proved to be completely integrable (cf. [O7] and [O9]). They are reduced to the Schrödinger operators with the potential functions $R(x_1, \dots, x_n)$ in the following list.

$$\begin{aligned}&\sum_{1\le i<j\le n} C_1\bigl(\sinh^{-2}\tfrac{x_i+x_j}{2} + \sinh^{-2}\tfrac{x_i-x_j}{2}\bigr)\\ &\qquad + \sum_{k=1}^{n}\bigl(C_2\sinh^{-2}x_k + C_3\sinh^{-2}\tfrac{x_k}{2}\bigr),\end{aligned} \qquad \text{(Trig-}BC_n\text{-reg)}$$

$$\sum_{1\le i<j\le n} C_1\sinh^{-2}\tfrac{x_i-x_j}{2} + \sum_{k=1}^{n}\bigl(C_2 e^{x_k} + C_3 e^{2x_k}\bigr), \qquad \text{(Trig-}A_{n-1}\text{-bry-reg)}$$

$$C_1\sum_{i=1}^{n-1} e^{x_i - x_{i+1}} + C_1 e^{x_{n-1}+x_n} + C_2\sinh^{-2}\tfrac{x_n}{2} + C_3\sinh^{-2}x_n, \qquad \text{(Toda-}D_n\text{-bry)}$$

$$C_1\sum_{i=1}^{n-1} e^{x_i-x_{i+1}} + C_2 e^{x_n} + C_3 e^{2x_n}. \qquad \text{(Toda-}BC_n\text{)}$$

Here C_1, C_2 and C_3 are any complex numbers.

We can generalize the Schrödinger operators in terms of root systems (cf. [OP]). Let Σ be an irreducible root system with rank n, Σ^+ a positive system of Σ and $\Psi \subset \Sigma$ a fundamental system of Σ^+. Then Σ is identified with a finite subset of a Euclidean space $\mathbb{R}^n$ and

$$P = \sum_{k=1}^{n}\frac{\partial^2}{\partial x_k^2} + \sum_{\alpha\in\Sigma^+}\frac{C_\alpha}{\sinh^2\frac{\langle\alpha,x\rangle}{2}} \quad (C_\alpha\in\mathbb{C},\ C_\alpha = C_\beta \text{ if } |\alpha| = |\beta|) \tag{28}$$

and

$$P = \sum_{k=1}^{n}\frac{\partial^2}{\partial x_k^2} + \sum_{\alpha\in\Psi} e^{\langle\alpha,x\rangle} \tag{29}$$

are Schrödinger operators of Heckman-Opdam's hypergeometric system (cf. [HO]) and Toda finite chain (cf. [To]) corresponding to the fundamental system Ψ, respectively. They are in $\mathcal{D}_*$ under the coordinate system

$$t_k = e^{\langle\alpha_k,x\rangle} \quad \text{for} \quad k = 1,\dots,n$$

with $\Psi = \{\alpha_1,\dots,\alpha_n\}$ and known to be completely integrable.

If Σ is of type BC_n, then

$$\Sigma^+ = \{e_i - e_j,\ e_k,\ 2e_k\,;\ 1\le i<j\le n,\ 1\le k\le n\}$$

and the Schrödinger operators (28) and (29) correspond to (Trig-BC_n-reg) or (Toda-BC_n). If Σ is of other classical type, the operators also correspond to special cases of (Trig-A_{n-1}-bry) or (Toda-D_n-bry) or (Toda-BC_n).

The potential functions $R(x)$ of known completely integral quantum systems which may not have regular singularities at infinity are expressed by functions with one variable. If P_2 and P_3 are operators of order 4 and 6 with the highest order terms $\sum_{k=1}^n \frac{\partial^4}{\partial x_k^4}$ and $\sum_{k=1}^n \frac{\partial^6}{\partial x_k^6}$, respectively, this is proved by [Wa] in general. We will examine this in the case when $n = 2$.

Theorem 8.1. *Let ℓ be a positive integer. Suppose the differential operators*

$$\begin{aligned} P &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + R(x,y), \\ Q &= \sum_{i=0}^m c_i \frac{\partial^m}{\partial x^{m-i}\partial y^i} + \sum_{i+j\le m-2} S_{i,j}(x,y)\frac{\partial^{i+j}}{\partial x^i \partial y^j} \end{aligned} \tag{30}$$

satisfy $[P,Q]=0$ and $\sigma_m(Q)\notin \mathbb{C}[\sigma(P)]$. Here $R(x,y)$ and $S_{i,j}(x,y)$ are square matrices of size ℓ whose components are functions of (x,y) and $c_i\in\mathbb{C}$. Put

$$\Big(\xi\frac{\partial}{\partial\tau} - \tau\frac{\partial}{\partial\xi}\Big)\sum_{i=0}^m c_i\xi^{m-i}\tau^i = \prod_{\nu=1}^L (a_\nu\xi - b_\nu\tau)^{m_\nu} \tag{31}$$

with suitable $(a_\nu,b_\nu)\in\mathbb{C}^2\setminus\{0\}$ satisfying $a_\nu b_\mu \ne a_\mu b_\nu$ for $\mu\ne\nu$. Here m_ν are positive integers and $m_1+\cdots+m_L=m$. Then

$$R(x,y) = \sum_{\nu=1}^L \sum_{i=0}^{m_\nu-1} (b_\nu x + a_\nu y)^i R_{\nu,i}(a_\nu x - b_\nu y) \tag{32}$$

with m square matrices of size ℓ whose components are functions $R_{\nu,i}(t)$ of the one variable t.

Proof. The coefficients of $\frac{\partial^{m+1}}{\partial x^{m-1-j}\partial y^j}$ in the expression $[P,Q]$ for (30) show

$$2\partial_x S_{m-2-j,j} + 2\partial_y S_{m-1-j,j-1} = c_j(m-j)\partial_x R + c_{j+1}(j+1)\partial_y R$$

for $j=0,\ldots,m-1$. Hence the theorem follows from the following equation.

$$\begin{aligned} 0 &= 2\sum_{j=0}^{m-1}(-1)^j\big(\partial_x^j\partial_y^{m-j}S_{m-2-j,j} + \partial_x^{j+1}\partial_y^{m-1-j}S_{m-1-j,j-1}\big) \\ &= 2\sum_{j=0}^{m-1}(-1)^j\partial_x^j\partial_y^{m-1-j}\big(c_j(m-j)\partial_x R + c_{j+1}(j+1)\partial_y R\big) \\ &= \sum_{j=0}^{m-1}(-1)^j c_j(m-j)\partial_x^{j+1}\partial_y^{m-1-j}R + \sum_{j=0}^{m-1}(-1)^j c_{j+1}(j+1)\partial_x^j\partial_y^{m-j}R \\ &= \Big((\xi\frac{\partial}{\partial\tau} - \tau\frac{\partial}{\partial\xi})\sum_{i=0}^m c_i\xi^{m-i}\tau^i\Big)\Big|_{\xi=\partial_y,\,\tau=-\partial_x} R. \end{aligned}$$

□

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