Quantization of linear algebra and its application to integral geometry

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Abstract. In order to construct good generating systems of two-sided ideals in the universal enveloping algebra of a complex reductive Lie algebra, we quantize some notions of linear algebra, such as minors, elementary divisors, and minimal polynomials. The resulting systems are applied to the integral geometry on various homogeneous spaces of related real Lie groups.

1. Introduction

When a real Lie group $G_{\mathbb{R}}$ acts on a homogeneous space, the space of functions or line bundle sections on the homogeneous space is naturally an infinite dimensional representation of $G_{\mathbb{R}}$. One knows many important representations are realized as subrepresentations of that kind of spaces. Here it is quite usual that those subrepresentations are characterized as the solutions of certain systems of differential equations. In the first half of this article, we explain many such systems of equations can be obtained through *a quantization* of elementary geometrical objects. For the most part our discussion is based on examples for $GL(n,\mathbb{C})$, where our differential equations are quantizations of some notions in linear algebra because the geometry of $GL(n,\mathbb{C})$ is directly linked to linear algebra. In the second half, we show these differential equations for $GL(n,\mathbb{C})$ are equally applicable to the integral geometry of each real form of $GL(n,\mathbb{C})$.

Let $\mathfrak{g}_{\mathbb{R}}$ be the Lie algebra of $G_{\mathbb{R}}$, $\mathfrak{g}$ its complexification, and $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. In general, the annihilator of a representation is a two-sided ideal in $U(\mathfrak{g})$. If G is the adjoint group of $\mathfrak{g}$ (or a connected complex Lie group with Lie algebra $\mathfrak{g}$), then a two-sided ideal in $U(\mathfrak{g})$ is a left ideal which is stable under the adjoint action of G. Hence, in the symmetric algebra $S(\mathfrak{g})$ of $\mathfrak{g}$, which is considered as the *classical limit* of $U(\mathfrak{g})$, a G-stable ideal is the classical counterpart of a two-sided ideal in $U(\mathfrak{g})$. Now suppose $S(\mathfrak{g})$ can be identified in a natural way with the algebra $P(\mathfrak{g})$ of polynomial functions on $\mathfrak{g}$. Thus to a conjugacy class of any $A \in \mathfrak{g}$ there corresponds a *big* G-stable ideal of $S(\mathfrak{g})$. We regard a certain primitive ideal in $U(\mathfrak{g})$ as a quantization of this ideal. Our systems of differential equations are some good generating systems of these primitive ideals.

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2. Conjugacy classes and scalar generalized Verma modules

For a while assume $G = GL(n,\mathbb{C})$. As usual, we relate an $n \times n$ matrix $A \in M(n,\mathbb{C})$ to the left invariant holomorphic vector field on G defined by $\varphi(x) \mapsto \frac{d}{dt}\varphi(xe^{tA})\big|_{t=0} = \frac{d}{dt}\varphi(x+txA)\big|_{t=0}$. The Lie algebra $\mathfrak{g} = \mathfrak{gl}_n$ of G is thus identified with $M(n,\mathbb{C})$. More explicitly, if $E_{ij} \in M(n,\mathbb{C})$ is the matrix with 1 in the (i,j) position and 0 elsewhere, the identification is written as

$$E_{ij} = \sum_{\nu=1}^{n} x_{\nu i}\frac{\partial}{\partial x_{\nu j}}.$$

Also, the adjoint action of $g \in G$ on $\mathfrak{g}$ reduces to $\operatorname{Ad}(g) : A \mapsto gAg^{-1}$. We denote the algebra automorphisms of $U(\mathfrak{g})$, $S(\mathfrak{g})$ and $P(\mathfrak{g})$ induced from $\operatorname{Ad}(g)$ by the same symbol. In this section we study an $\operatorname{Ad}(G)$-stable ideal in $U(\mathfrak{g})$ which is considered as a quantization of the defining ideal for the conjugacy class $V_A = \bigcup_{g\in G}\operatorname{Ad}(g)A$ (or its closure $\overline{V}_A$).

Using the nondegenerate symmetric bilinear form

$$\langle X, Y\rangle = \operatorname{Trace} XY, \tag{2.1}$$

we identify $\mathfrak{g}$ with its dual space $\mathfrak{g}^*$, and $S(\mathfrak{g})$ with $P(\mathfrak{g}) = S(\mathfrak{g}^*)$. The following scheme shows our standpoint:

$$\begin{array}{ccc}
V_A = \bigcup_{g\in G}\operatorname{Ad}(g)A & \longrightarrow & (G\text{-stable) defining ideal of } \overline{V}_A \\
\vdots & & \Big\downarrow \text{quantization} \\
\text{rep's of } U(\mathfrak{g}) \text{ or a real form } G_{\mathbb{R}} \text{ of } G & \longleftarrow & G\text{-stable ideal of } U(\mathfrak{g})
\end{array}$$

In order to study the classical object $S(\mathfrak{g})$ and its quantization $U(\mathfrak{g})$ at one time, the notion of *homogenized enveloping algebra* was introduced by [**Os4**]. It is an algebra defined by

$$U^{\epsilon}(\mathfrak{g}) := \left(\mathbb{C}[\epsilon]\otimes\sum_{m=0}^{\infty}\bigotimes^{m}\mathfrak{g}\right)\Big/\Big\langle X\otimes Y - Y\otimes X - \epsilon[X,Y];\ X,Y\in\mathfrak{g}\Big\rangle. \tag{2.2}$$

Here ϵ is a complex number or an indeterminant which commutes with all elements. Clearly $U(\mathfrak{g}) = U^1(\mathfrak{g})$, $S(\mathfrak{g}) = U^0(\mathfrak{g})$. If $\epsilon \in \mathbb{C}^{\times}$ then the map $\mathfrak{g} \ni X \mapsto \epsilon^{-1}X \in U^{\epsilon}(\mathfrak{g})$ extends to an algebra isomorphism of $U^1(\mathfrak{g})$ onto $U^{\epsilon}(\mathfrak{g})$. On the other hand, when ϵ is an indeterminant, a choice of Poincaré-Birkhoff-Witt basis naturally induces an isomorphism $U(\mathfrak{g})\otimes\mathbb{C}[\epsilon] \xrightarrow{\sim} U^{\epsilon}(\mathfrak{g})$ of linear spaces. Furthermore, since the generators of the denominator of (2.2) are homogeneous of degree 2 with respect to ϵ and $X \in \mathfrak{g}$, we can endow $U^{\epsilon}(\mathfrak{g})$ with a graded algebra structure such that ϵ as well as any $X \in \mathfrak{g}$ has degree 1.

For a sequence $\{n'_1, \ldots, n'_L\}$ of positive integers whose sum is n, put

$$\begin{cases} n_k = n'_1 + \ldots + n'_k \quad (1 \le k \le L), \qquad n_0 = 0,\\ \Theta = \{n_1, \ldots, n_L\},\\ \iota_\Theta(\nu) = k \ \text{ if } \ n_{k-1} < \nu \le n_k \quad (1 \le k \le L). \end{cases}$$

Clearly Θ is a strictly increasing sequence of positive integers terminating at n and to such a sequence Θ there corresponds a unique $\{n'_1, \ldots, n'_L\}$. Let us define some

Lie subalgebras of $\mathfrak{g} = \mathfrak{gl}_n$ as follows:

$$\begin{aligned}
&\mathfrak{n} = \sum_{i>j} \mathbb{C}E_{ij}, && \bar{\mathfrak{n}} = \sum_{i<j} \mathbb{C}E_{ij}, && \mathfrak{a} = \sum_{i} \mathbb{C}E_{ii},\\
&\mathfrak{b} = \mathfrak{a} + \mathfrak{n}, && \mathfrak{n}_\Theta = \sum_{\iota_\Theta(i)>\iota_\Theta(j)} \mathbb{C}E_{ij}, && \bar{\mathfrak{n}}_\Theta = \sum_{\iota_\Theta(i)<\iota_\Theta(j)} \mathbb{C}E_{ij},\\
&\mathfrak{m}_\Theta = \sum_{\iota_\Theta(i)=\iota_\Theta(j)} \mathbb{C}E_{ij}, && \mathfrak{m}_\Theta^k = \sum_{\iota_\Theta(i)=\iota_\Theta(j)=k} \mathbb{C}E_{ij}, && \mathfrak{b}_\Theta = \mathfrak{m}_\Theta + \mathfrak{n}_\Theta.
\end{aligned}$$

One knows $\mathfrak{b}_\Theta$ is a standard parabolic subalgebra containing the Borel subalgebra $\mathfrak{b}$ and any standard parabolic subalgebra equals $\mathfrak{b}_\Theta$ for some unique Θ. Notice that $\mathfrak{m}_\Theta = \bigoplus_{k=1}^{L} \mathfrak{m}_\Theta^k$ and $\mathfrak{b}_\Theta = \{X \in \mathfrak{g};\ \langle X, Y\rangle = 0\,(\forall Y \in \mathfrak{n}_\Theta)\}$.

For a fixed $\lambda = (\lambda_1, \dots, \lambda_L) \in \mathbb{C}^L$, let us consider the affine subspace

$$\begin{aligned}
A_{\Theta,\lambda} &:= \sum_{i=1}^{n} \lambda_{\iota_\Theta(i)} E_{ii} + \mathfrak{n}_\Theta\\
&= \left\{ \begin{pmatrix} \lambda_1 I_{n'_1} & & & & \text{\Large 0}\\ A_{21} & \lambda_2 I_{n'_2} & & & \\ A_{31} & A_{32} & \lambda_3 I_{n'_3} & & \\ \vdots & \vdots & \vdots & \ddots & \\ A_{L1} & A_{L2} & A_{L3} & \cdots & \lambda_L I_{n'_L} \end{pmatrix} ;\ A_{ij} \in M(n'_i, n'_j; \mathbb{C}) \right\}
\end{aligned}$$

of $\mathfrak{g}$. Here I_m is the identity matrix of size m and $M(k, \ell; \mathbb{C})$ is the set of $k \times \ell$ matrices.

Remark 2.1. A generic element of $A_{\Theta,\lambda}$ belongs to a common conjugacy class, whose Jordan normal form is given by

$$\bigoplus_{\mu\in\mathbb{C},\,1\le k\le n} J\bigl(\#\{i;\ \lambda_i = \mu \text{ and } n'_i \ge k\}, \mu\bigr)$$

$$\text{where } J(m,\mu) = \begin{pmatrix} \mu & & & \text{\Large 0}\\ 1 & \mu & & \\ & \ddots & \ddots & \\ \text{\Large 0} & & 1 & \mu \end{pmatrix} \in M(m, \mathbb{C}).$$

Hereafter this conjugacy class is referred to as *the conjugacy class of* $A_{\Theta,\lambda}$. Any Jordan normal form is that of such a conjugacy class for some choice of Θ and λ. The closure of the conjugacy class of $A_{\Theta,\lambda}$ is

$$V_{A_{\Theta,\lambda}} := \bigcup_{g\in G} \operatorname{Ad}(g) A_{\Theta,\lambda}.$$

In the classical case, the condition that a function $f \in P(\mathfrak{g}) = S(\mathfrak{g}) = U^0(\mathfrak{g})$ vanishes on the conjugacy class of $A_{\Theta,\lambda}$ is equivalent to any of the following with $\epsilon = 0$:

$$\begin{aligned}
f(V_{A_{\Theta,\lambda}}) = \{0\} &\iff \bigl(\operatorname{Ad}(g) f\bigr)(A_{\Theta,\lambda}) = \{0\} \quad (\forall g \in G)\\
&\iff \operatorname{Ad}(g) f \in J_\Theta^\epsilon(\lambda) \quad (\forall g \in G)\\
&\iff f \in \bigcap_{g\in G} \operatorname{Ad}(g) J_\Theta^\epsilon(\lambda)\\
&\iff f \in \operatorname{Ann}_G\bigl(M_\Theta^\epsilon(\lambda)\bigr).
\end{aligned} \tag{2.3}$$

Here for $\forall\epsilon\in\mathbb{C}$ we set

$$\begin{aligned}
J^\epsilon_\Theta(\lambda) &:= \sum_{k=1}^{L}\sum_{X\in\mathfrak{m}^k_\Theta} U^\epsilon(\mathfrak{g})(X-\lambda_k\operatorname{Trace}(X)) + U^\epsilon(\mathfrak{g})\mathfrak{n}_\Theta,\\
M^\epsilon_\Theta(\lambda) &:= U^\epsilon(\mathfrak{g})/J^\epsilon_\Theta(\lambda),\\
\operatorname{Ann}\bigl(M^\epsilon_\Theta(\lambda)\bigr) &:= \bigl\{D\in U^\epsilon(\mathfrak{g});\ DM^\epsilon_\Theta(\lambda)=0\bigr\}\\
I^\epsilon_\Theta(\lambda) := \operatorname{Ann}_G\bigl(M^\epsilon_\Theta(\lambda)\bigr) &:= \bigl\{D\in U^\epsilon(\mathfrak{g});\ \operatorname{Ad}(g)D\in\operatorname{Ann}\bigl(M^\epsilon_\Theta(\lambda)\bigr)\ (\forall g\in G)\bigr\}.
\end{aligned}$$

When $\epsilon=1$ we omit the superscript 1 and use such notation as $M_\Theta(\lambda)=M^1_\Theta(\lambda)$. Also, when $\Theta=\{1,\ldots,n\}$ we omit the subscript Θ and use such notation as $M^\epsilon(\lambda)=M^\epsilon_\Theta(\lambda)$. $M_\Theta(\lambda)$ is called *a scalar generalized Verma module* and is a quotient $\mathfrak{g}$-module of the *Verma module* $M(\lambda_\Theta)$ for the parameter

$$\lambda_\Theta := (\underbrace{\lambda_1,\ldots,\lambda_1}_{n'_1},\underbrace{\lambda_2,\ldots,\lambda_2}_{n'_2},\ldots,\underbrace{\lambda_L,\ldots,\lambda_L}_{n'_L})\in\mathbb{C}^n. \tag{2.4}$$

Since we have realized that the defining ideal of $V_{A_{\Theta,\lambda}}$ is $I^0_\Theta(\lambda)=\operatorname{Ann}_G\bigl(M^\epsilon_\Theta(\lambda)\bigr)$, it is natural to think its quantization is $I_\Theta(\lambda)=\operatorname{Ann}_G\bigl(M_\Theta(\lambda)\bigr)=\operatorname{Ann}\bigl(M_\Theta(\lambda)\bigr)$. In fact the last two equivalences in (2.3) are valid for any $\epsilon\in\mathbb{C}$ and any $f\in U^\epsilon(\mathfrak{g})$.

Now let us formulate the main problem in the first half of this article.

PROBLEM 2.2. For $\epsilon=0,1$ construct good generating systems of $I^\epsilon_\Theta(\lambda)$.

In the following sections we shall give some concrete answers. Our generating systems will always be in $U^\epsilon(\mathfrak{g})$ and they are valid for any ϵ.

3. Eigenvalues and determinants

The space $\mathfrak{a}=\sum_{i=1}^n\mathbb{C}E_{ii}$ of diagonal matrices is isomorphic to $\mathbb{C}^n=\{(x_1,\ldots,x_n)\}$ on which the n-th symmetric group $\mathfrak{S}_n$ acts by permutation of coordinates. If we identify $S(\mathfrak{a})$ with $P(\mathfrak{a})$ by (2.1) then the restriction map $S(\mathfrak{g})\to S(\mathfrak{a})$ is naturally defined and *the Chevalley restriction theorem* asserts it induces the algebra isomorphism

$$\Gamma^0 : S(\mathfrak{g})^G \simeq S(\mathfrak{a})^{\mathfrak{S}_n}.$$

One knows the elementary symmetric polynomials $s_m(x)=\sum_{1\le i_1<\cdots<i_m\le m}x_{i_1}\cdots x_{i_m}$ ($m=1,\ldots,n$) generate $S(\mathfrak{a})^{\mathfrak{S}_n}$ and so do the power sum polynomials $S_m(x)=\sum_{i=1}^n x_i^m$ ($m=1,2,\ldots$).

The *eigenvalues* of any matrix in $V_{A_{\Theta,\lambda}}$, counted with multiplicities, coincide with the entries of λ_Θ given by (2.4). Thus the collection of them is an invariant of $V_{A_{\Theta,\lambda}}$. We note it is completely determined by the values at λ_Θ of the elements in a generating systems of $S(\mathfrak{a})^{\mathfrak{S}_n}$, e.g. the sequence $\{s_1(\lambda_\Theta),\ldots,s_n(\lambda_\Theta)\}$, or the sequence $\{S_1(\lambda_\Theta),S_2(\lambda_\Theta),\ldots\}$. Now any $f\in S(\mathfrak{g})^G$ takes the value $\Gamma^0(f)(\lambda_\Theta)$ constantly on $V_{A_{\Theta,\lambda}}$. Analogously, any $D\in U(\mathfrak{g})^G$ acts on $M_\Theta(\lambda)$ by a scalar. (Namely, $M_\Theta(\lambda)$ has an *infinitesimal character*.) These are special cases of the general fact that $D\in U^\epsilon(\mathfrak{g})^G$ acts on $M^\epsilon_\Theta(\lambda)$ by the scalar $\gamma^\epsilon(D)(\lambda_\Theta)$. Here γ^ϵ denotes the *quantization* of the restriction map $S(\mathfrak{g})\to S(\mathfrak{a})$ defined by

$$\gamma^\epsilon : U^\epsilon(\mathfrak{g})\ni D\mapsto\gamma^\epsilon(D)\in U^\epsilon(\mathfrak{a})=S(\mathfrak{a})\qquad\bigl(D-\gamma^\epsilon(D)\in\bar{\mathfrak{n}}U^\epsilon(\mathfrak{g})+U^\epsilon(\mathfrak{g})\mathfrak{n}\bigr).$$

Note $U^\epsilon(\mathfrak{g})=\bigl(\bar{\mathfrak{n}}U^\epsilon(\mathfrak{g})+U^\epsilon(\mathfrak{g})\mathfrak{n}\bigr)\oplus U^\epsilon(\mathfrak{a})$ is a direct sum decomposition and $U^\epsilon(\mathfrak{a})=S(\mathfrak{a})$ by the commutativity of $\mathfrak{a}$. If we put $\rho=\sum_{i=1}^n(i-\frac{n+1}{2})E_{ii}$ and define the translation $T_{\epsilon\rho} : S(\mathfrak{a}) \ni D \mapsto D(\cdot - \epsilon\rho) \in S(\mathfrak{a})$ under the identification $S(\mathfrak{a}) = P(\mathfrak{a})$, then $\Gamma^\epsilon := T_{\epsilon\rho} \circ \gamma^\epsilon$ induces the algebra isomorphism

$$\Gamma^\epsilon : U^\epsilon(\mathfrak{g})^G \simeq S(\mathfrak{a})^{\mathfrak{S}_n}. \tag{3.1}$$

If $\epsilon \neq 0$ then (3.1) is the celebrated *Harish-Chandra isomorphism*. So we refer to γ^ϵ or Γ^ϵ as the *Harish-Chandra map*.

Put $\mathbb{E} = \big(E_{ij}\big) \in M(n, U^\epsilon(\mathfrak{g}))$. Since

$$\mathrm{Ad}(g)\mathbb{E} = {}^t g \, \mathbb{E} \, {}^t g^{-1} \qquad (\forall g \in G), \tag{3.2}$$

we have

$$Z_m := \mathrm{Trace}\, \mathbb{E}^m \qquad (m = 1, 2, \dots) \tag{3.3}$$

in $U^\epsilon(\mathfrak{g})^G$ (cf. Gelfand's construction in [**Ge**]). It is easy to see that the highest homogeneous part of $\Gamma^\epsilon(Z_m)$ equals $S_m(x)$. Hence $U^\epsilon(\mathfrak{g})^G = \mathbb{C}[Z_1, \dots, Z_n]$ ($\forall \epsilon \in \mathbb{C}$). Although the equality $\Gamma^0(Z_m) = S_m(x)$ is immediate, a nontrivial calculation is necessary to write $\Gamma^1(Z_m)$ down explicitly (cf. §7 Remark 7.2 ii)).

Now, for $t \in \mathbb{C}$ we define a *quantized determinant* by

$$D(t) := \det\Big(E_{ij} + \big(\epsilon(n-j) - t\big)\delta_{ij}\Big) \in U^\epsilon(\mathfrak{g}), \tag{3.4}$$

where we suppose the determinant in the right-hand side is a so-called *column determinant*. Throughout the article, the determinant of a square matrix $\big(A_{ij}\big)$ with non-commutative entries means the column determinant given by

$$\det\big(A_{ij}\big) = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) A_{\sigma(1)1} \cdots A_{\sigma(n)n}.$$

The G-invariance of $D(t)$ with $\epsilon = 1$ is a well-known classical result (cf. [**Ca1**]), which in fact follows from the *Capelli identity*

$$\det\big(E_{ij} + (n-j)\delta_{ij}\big) = \det\big(x_{ij}\big) \det\Big(\frac{\partial}{\partial x_{ij}}\Big).$$

and the algebra automorphism of $U(\mathfrak{g})$ defined by $E_{ij} \mapsto E_{ij} - t\delta_{ij}$. More generally we have $D(t) \in U^\epsilon(\mathfrak{g})^G$. The image of $D(t)$ under the Harish-Chandra map is easily calculated:

$$\gamma^\epsilon(D(t)) = \prod_{i=1}^n \big(E_{ii} - t + \epsilon(i-1)\big), \quad \Gamma^\epsilon(D(t)) = \sum_{m=0}^n s_m(x) \Big(\frac{n-1}{2}\epsilon - t\Big)^{n-m}.$$

(Here we let $s_0(x) = 1$.) Hence if we consider $D(t) \in U^\epsilon(\mathfrak{g})^G[t]$ and denote its coefficient of t^m by Δ_m, then $U^\epsilon(\mathfrak{g})^G = \mathbb{C}[\Delta_0, \dots, \Delta_{n-1}]$ ($\forall \epsilon \in \mathbb{C}$).

REMARK 3.1. i) When ϵ is an indeterminant, $U^\epsilon(\mathfrak{g})^G = \mathbb{C}[\epsilon, Z_1, \dots, Z_n] = \mathbb{C}[\epsilon, \Delta_0, \dots, \Delta_{n-1}]$.
ii) Various relations between $\{Z_m\}$, $D(t)$ and other central elements in $U(\mathfrak{g})$ are studied by T. Umeda [**U**] and M. Ito [**I**].
iii) The construction method (3.3) of central elements applies to general complex reductive Lie algebras (cf. §7). On the other hand, there is a version of (3.4) for $\mathfrak{g} = \mathfrak{o}_n$, the Lie algebra of $O(n, \mathbb{C})$ (cf. [**HU**], [**Wa**]).

Suppose $\lambda \in \mathbb{C}^n$. Since $M^\epsilon(\lambda)$ has infinitesimal character $U^\epsilon(\mathfrak{g})^G \ni D \mapsto \gamma^\epsilon(D)(\lambda) \in \mathbb{C}$, we have $D - \gamma^\epsilon(D)(\lambda) \in I^\epsilon(\lambda)$ $(\forall D \in U^\epsilon(\mathfrak{g})^G)$. More strongly,

$$I^\epsilon(\lambda) = \sum_{D \in U^\epsilon(\mathfrak{g})^G} U^\epsilon(\mathfrak{g})\big(D - \gamma^\epsilon(D)(\lambda)\big). \tag{3.5}$$

When $\lambda = 0$ and $\epsilon = 0$, the above equality reduces to the assertion that the defining ideal of $V_{A_{\{1,\dots,n\},0}}$ is generated by the G-invariant polynomials without constant term. Here we note $V_{A_{\{1,\dots,n\},0}}$ is the set of all nilpotent matrices. This assertion is proved by B. Kostant [**Ko**] for all complex reductive Lie algebras, from which (3.5) in the general case is readily deduced.

For $\lambda \in \mathbb{C}^n$ put

$$\mathcal{I}^\epsilon_\lambda := \big\{D \in U^\epsilon(\mathfrak{g});\ \gamma^\epsilon(\mathrm{Ad}(g)D)(\lambda) = 0\ (\forall g \in G)\big\}.$$

It is a two-sided ideal of $U^\epsilon(\mathfrak{g})$. If $\epsilon = 0$ then $\mathcal{I}^0_\lambda$ is the defining ideal of the conjugacy class V_λ of a diagonalizable matrix whose eigenvalues are the entries of λ. If $\epsilon = 1$ then $\mathcal{I}_\lambda = \mathrm{Ann}\big(L(\lambda)\big) := \big\{D \in U(\mathfrak{g});\ DL(\lambda) = 0\big\}$ where $L(\lambda)$ is the unique irreducible quotient of the Verma module $M(\lambda)$. Thus $\mathcal{I}_\lambda$ is a primitive ideal of $U(\mathfrak{g})$. Conversely, $\{\mathcal{I}_\lambda;\ \lambda \in \mathbb{C}^n\}$ equals the set of all primitive ideals (cf. [**Du**]). For $w \in \mathfrak{S}_n$ define its *shifted action* by $w.\lambda = w(\lambda + \epsilon\rho) - \epsilon\rho$. Then it holds that $\mathcal{I}^\epsilon_{w.\lambda} = \mathcal{I}^\epsilon_\lambda$ for a generic λ. This is not true for some λ (for example, when $\lambda = 0$).

Now suppose Θ, $\lambda \in \mathbb{C}^{\#\Theta}$ are arbitrary. Since $M^\epsilon_\Theta(\lambda)$ is a quotient of $M^\epsilon(\lambda_\Theta)$, $I^\epsilon_\Theta(\lambda) \supset I^\epsilon(\lambda_\Theta)$. On the other hand, we assert $I^\epsilon_\Theta(\lambda) \subset \mathcal{I}^\epsilon_{\lambda_\Theta}$ and the equality holds for a generic $\lambda \in \mathbb{C}^{\#\Theta}$. In fact, when $\epsilon = 0$ we have $V_{A_{\Theta,\lambda}} \supset V_{\lambda_\Theta}$ and $V_{A_{\Theta,\lambda}} = V_{\lambda_\Theta}$ if each entry of $\lambda \in \mathbb{C}^{\#\Theta}$ is distinct. So $I^0_\Theta(\lambda) \subset \mathcal{I}^0_{\lambda_\Theta}$ and both are equal for a generic λ. When $\epsilon = 1$, since $L(\lambda_\Theta)$ is the unique irreducible quotient of $M_\Theta(\lambda)$ and since $M_\Theta(\lambda)$ is irreducible for a generic λ, the assertion holds. Finally we remark $I_\Theta(\lambda)$ is always a primitive ideal because even if $M_\Theta(\lambda)$ is reducible, we can choose a $w \in \mathfrak{S}_n$ so that $I_\Theta(\lambda) = \mathcal{I}_{w.\lambda_\Theta}$.

4. Restriction to the diagonal part and completely integrable quantum systems

When one wants to construct a generating system of the defining ideal $\mathcal{I}^0_\lambda$ of a semisimple conjugacy class V_λ $(\lambda \in \mathbb{C}^n)$, it is very useful to consider the restriction of $\mathcal{I}^0_\lambda$ to the diagonal part $\mathfrak{a}$, that is, the ideal of $S(\mathfrak{a}) = \mathbb{C}[x_1, \dots, x_n]$ defined by

$$\gamma^0\big(\mathcal{I}^0_\lambda\big) := \big\{\gamma^0(f) = f|_{\mathfrak{a}};\ f \in I^0_\lambda\big\}.$$

Let $\mathcal{I}(\mathfrak{S}_n\lambda)$ denote the defining ideal of the finite subset $\mathfrak{S}_n\lambda$ in $\mathbb{C}^n$.

Lemma 4.1. *Suppose $\widetilde{\mathcal{E}} \subset S(\mathfrak{g})$ is a G-stable linear subspace. Then $\mathcal{E} := \gamma^0\big(\widetilde{\mathcal{E}}\big)$ is an $\mathfrak{S}_n$-stable linear subspace of $S(\mathfrak{a})$ and*

$$\widetilde{\mathcal{E}} \text{ generates } \mathcal{I}^0_\lambda \Longleftrightarrow \mathcal{E} \text{ generates } \mathcal{I}(\mathfrak{S}_n\lambda).$$

In particular $\gamma^0\big(\mathcal{I}^0_\lambda\big) = \mathcal{I}(\mathfrak{S}_n\lambda)$.

For example, $\mathcal{I}(\mathfrak{S}_n\lambda)$ contains

$$s_m(x) - s_m(\lambda) \qquad (m = 1, \dots, n). \tag{4.1}$$

If each entry of $\lambda = (\lambda_1, \dots, \lambda_n)$ is distinct, then $\mathfrak{S}_n\lambda$ consists of $n!$ points and (4.1) generates $\mathcal{I}(\mathfrak{S}_n\lambda)$ (so does $\{S_m(x) - S_m(\lambda);\ m = 1, \dots, n\}$). In this case, $\mathcal{I}^0_\lambda = I^0(\lambda)$ and by Lemma 4.1 the assertion above is equivalent to (3.5) with $\epsilon = 0$.

In addition to (4.1), $\mathcal{I}(\mathfrak{S}_n\lambda)$ contains

$$\prod_{i=1}^{n}(x_i-\lambda_j) \qquad (j=1,\dots,n), \tag{4.2}$$

$$\prod_{j=1}^{n}(x_i-\lambda_j) \qquad (i=1,\dots,n). \tag{4.3}$$

If each entry of λ is distinct, (4.2) is also a generating system. But in other cases, even the combination of (4.2) with (4.3) does not generate $\mathcal{I}(\mathfrak{S}_n\lambda)$. Suppose $\lambda = (\underbrace{\mu,\dots,\mu}_{k},\underbrace{\nu,\dots,\nu}_{n-k})$, $\mu \neq \nu$, for example. In this case $\mathfrak{S}_n\lambda$ consists of $\frac{n!}{k!(n-k)!}$ points and instead of (4.2) or (4.3), we should consider the following elements in $\mathcal{I}(\mathfrak{S}_n\lambda)$:

$$\begin{cases} (x_{i_1}-\mu)\cdots(x_{i_{n-k+1}}-\mu) & (1 \le i_1 < \cdots < i_{n-k+1} \le n), \\ (x_{j_1}-\nu)\cdots(x_{j_{k+1}}-\nu) & (1 \le j_1 < \cdots < j_{k+1} \le n), \end{cases} \tag{4.4}$$

$$(x_i-\mu)(x_i-\nu) \qquad (i=1,\dots,n). \tag{4.5}$$

Then (4.4) generates $\mathcal{I}(\mathfrak{S}_n\lambda)$. Also, the system (4.5), together with (4.1) for $m = 1$, generates $\mathcal{I}(\mathfrak{S}_n\lambda)$. Note that both (4.4) and (4.5) span G-stable linear subspaces.

Remark 4.2. The *$\mathfrak{S}_n$-invariant Completely integrable quantum systems* are those systems of differential equations on $\mathfrak{a}$ which are classified by [**OSe**]. They are considered as quantizations of (4.1) (cf. [**OP**], [**Os5**]). In general, solutions (wave functions) of these systems are not so well understood. But on the Heckman-Opdam hypergeometric functions (cf. [**HO**]) and on the generalized Bessel functions (cf. [**Op**]), there are many results. The most trivial way of quantization is to simply replace x_i with $\frac{\partial}{\partial x_i}$ in (4.1). The solution space of this system is spanned by exponential polynomials and as an $\mathfrak{S}_n$-module it is isomorphic to the regular representation of $\mathfrak{S}_n$. When $\lambda = 0$ a solution of the system is a so-called $\mathfrak{S}_n$-harmonic polynomial. A basis of the solution space which is entirely holomorphic in (x,λ) is given by [**Os2**].

Remark 4.3. For any $\mathfrak{S}_n$-stable linear subspace $\mathcal{E} \subset S(\mathfrak{a})$ there exists a $GL(n,\mathbb{C})$-stable linear subspace $\widetilde{\mathcal{E}} \subset S(\mathfrak{g})$ such that $\mathcal{E} = \gamma^0(\widetilde{\mathcal{E}})$. But the corresponding assertion is not always true for a general complex reductive Lie group or in the similar setting for a Riemannian symmetric space (cf. [**Br**], [**Od4**]).

5. Minors

The *rank* of a matrix is also a basic invariant of a conjugacy class. Recall it is described in terms of the *minors*. For example, suppose $\Theta = \{k,n\}$ and $\lambda = (\mu,\nu)$. Thus

$$A_{\Theta,\lambda} = \left\{ \begin{pmatrix} \mu I_k & 0 \\ * & \nu I_{n-k} \end{pmatrix} \right\}, \tag{5.1}$$

and for any $A \in V_{A_{\Theta,\lambda}}$ we have $\operatorname{rank}(A-\mu) \le n-k$ and $\operatorname{rank}(A-\nu) \le k$. Hence the minors of $\big(E_{ij}-\mu\big) \in M(n,S(\mathfrak{g}))$ with size $n-k+1$ and the minors of $\big(E_{ij}-\nu\big) \in M(n,S(\mathfrak{g}))$ with size $k+1$ vanish on $V_{A_{\Theta,\lambda}}$.

For $t \in \mathbb{C}$ let us define a *quantization of the minors* of $\bigl(E_{ij} - t\bigr)$ to be

$$D_{\{i_1,\ldots,i_m\}\{j_1,\ldots,j_m\}}(t) := \det\Bigl(E_{i_p j_q} + \bigl(\epsilon(m-q) - t\bigr)\delta_{i_p j_q}\Bigr)_{\substack{1\le p\le m\\ 1\le q\le m}} \in U^\epsilon(\mathfrak{g})$$

$$\Bigl(\{i_1,\ldots,i_m\},\{j_1,\ldots,j_m\} \subset \{1,\ldots,n\}\Bigr)$$

and call them the *generalized Capelli elements*. As in the classical case, they change their sign by a transposition of row or column indices, and for any fixed t and ϵ, $\bigl\{D_{IJ}(t);\, \#I = \#J = m,\, I, J \subset \{1,\ldots,n\}\bigr\}$ span a G-stable linear space. Moreover, if $\epsilon = 1$ we have the following *generalized Capelli identity*:

$$(5.2)\quad D_{\{i_1,\ldots,i_m\}\{j_1,\ldots,j_m\}}(0) = \sum_{1\le\nu_1<\cdots<\nu_m\le n} \det\bigl(x_{\nu_p i_q}\bigr)_{\substack{1\le p\le m\\ 1\le q\le m}} \cdot \det\Bigl(\frac{\partial}{\partial x_{\nu_p j_q}}\Bigr)_{\substack{1\le p\le m\\ 1\le q\le m}}.$$

Now suppose $\Theta = \{k, n\}$ and $\lambda = (\mu,\nu)$ again. Then for any ϵ

$$(5.3)\qquad D_{IJ}(\mu),\, D_{I'J'}(\nu + k\epsilon) \qquad (\#I = \#J = n-k+1,\, \#I' = \#J' = k+1)$$

belong to $I^\epsilon_\Theta(\lambda)$. It can be shown by calculating their images under the Harish-Chandra map γ^ϵ. In the classical case where $\epsilon = 0$, because γ^0 maps (5.3) to (4.4), Lemma 4.1 implies that if $\mu \ne \nu$ then (5.3) generate $I^0_\Theta(\lambda) = \mathcal{I}^0_{\lambda_\Theta}$. For a general ϵ, we can show (5.3) generate $I^\epsilon_\Theta(\lambda)$ if $\mu - \nu \notin \{\epsilon, 2\epsilon, \ldots, (n-1)\epsilon\}$. In order to obtain a generating system of $I^\epsilon_\Theta(\lambda)$ for any case, including the case where $\epsilon = 0$ and $\mu = \nu$, it is not sufficient to consider only the generalized Capelli elements. Besides them, we need the notion of *elementary divisors* and their quantization, which are discussed in the next section.

Remark 5.1. i) For any Θ and a generic $\lambda \in \mathbb{C}^{\#\Theta}$ we can construct a generating system of $I^\epsilon_\Theta(\lambda)$ which consists only of generalized Capelli elements (see (6.1)).
ii) When $\mathfrak{g} = \mathfrak{o}_n$, we can use a suitable quantization of the minor Pfaffians and the minor versions of the quantized determinant given by Howe–Umeda [**HU**] to construct a generating system of (the corresponding object to) $I^\epsilon_\Theta(\lambda)$ (cf. [**Od1**], [**Od2**]).

6. Elementary divisors

Let $\mathfrak{g} = \mathfrak{gl}_n$ and suppose $\Theta, \lambda \in \mathbb{C}^L$ $(L = \#\Theta)$ and $\epsilon \in \mathbb{C}$ are arbitrary.

Definition 6.1 ([**Os4**]). For $m = 1,\ldots,n$ define

$$\begin{cases} d^\epsilon_m(t;\Theta,\lambda) := \displaystyle\prod_{k=1}^{L} (t - \lambda_k - \epsilon n_{k-1})^{(n'_k + m - n)}, \\ d_m(\Theta) := \deg_t d^\epsilon_m(t;\Theta,\lambda) = \displaystyle\sum_{k=1}^{L} \max\{n'_k + m - n, 0\}, \\ e^\epsilon_m(t;\Theta,\lambda) := d^\epsilon_m(t;\Theta,\lambda)/d^\epsilon_{m-1}(t;\Theta,\lambda). \end{cases}$$

Here $d^\epsilon_0(t;\Theta,\lambda) = 1$ and

$$z^{(i)} := \begin{cases} z\bigl(z-\epsilon\bigr)\cdots\bigl(z - \epsilon(i-1)\bigr) & \text{if } i > 0, \\ 1 & \text{if } i \le 0. \end{cases}$$

We call $\{e^\epsilon_m(t;\Theta,\lambda);\, 1 \le m \le n\}$ the *elementary divisors* of $M^\epsilon_\Theta(\lambda)$.

If $\epsilon = 0$ and A is a generic in $A_{\Theta,\lambda}$ then $d^0_m(t;\Theta,\lambda)$ is the greatest common divisor of the minors of $tI_n - A$ with size m. Hence $e^0_m(t;\Theta,\lambda)$ is nothing but the mth elementary divisor of a generic element of $A_{\Theta,\lambda}$ in the sense of linear algebra.

Theorem 6.2 ([**Os4**]). *Write* $d^\epsilon_m(t;\Theta,\lambda) = \prod_{i=1}^{\ell_m}(t-\lambda_{m,i})^{N_{m,i}}$ $(i \neq i' \Rightarrow \lambda_{m,i} \neq \lambda_{m,i'})$ *and put*

$$\widetilde{\mathcal{E}}^\epsilon_\Theta(\lambda) := \sum_{m=1}^{n}\sum_{i=1}^{\ell_m}\sum_{j=0}^{N_{m,i}-1}\sum_{\#I=\#J=m}\mathbb{C}\left(\frac{d^j}{dt^j}D_{IJ}(t)\Big|_{t=\lambda_{m,i}}\right).$$

Then $I^\epsilon_\Theta(\lambda) = U^\epsilon(\mathfrak{g})\widetilde{\mathcal{E}}^\epsilon_\Theta(\lambda)$. *Moreover if all the roots of* $d^\epsilon_n(t;\Theta,\lambda)$ *are simple (in other words if* $w.\lambda_\Theta$ $(w \in \mathfrak{S}_n)$ *are all distinct), it holds that*

$$I^\epsilon_\Theta(\lambda) = \sum_{k=1}^{L}\sum_{\#I=\#J=n-n'_k+1} U^\epsilon(\mathfrak{g})D_{IJ}(\lambda_k + \epsilon n_{k-1}). \tag{6.1}$$

Remark 6.3. i) An inclusion relation between annihilator ideals reduces to a divisibility relation between the elementary divisors as follows:

$$I^\epsilon_\Theta(\lambda) \subset I^\epsilon_{\Theta'}(\lambda') \Longleftrightarrow d^\epsilon_m(t;\Theta,\lambda)\,|\,d^\epsilon_m(t;\Theta',\lambda') \qquad (m = 1,\ldots,n).$$

If $\epsilon = 0$ then the left-hand side is equivalent to the closure relation $V_{A_{\Theta,\lambda}} \supset V_{A_{\Theta',\lambda'}}$. In particular if $\epsilon = 0, \lambda = 0$ then it is a closure relation between conjugacy classes of nilpotent matrices (*nilpotent orbits*), which is equivalent to the well-known condition $d_m(\Theta) \le d_m(\Theta')$ $(m = 1,\ldots,n)$.
ii) The special case of Theorem 6.2 where $\epsilon = 0, \lambda = 0$ is conjectured by T. Tanisaki [**Ta1**] and is proved by J. Weymann [**We**]. Theorem 6.2 in the general case can be considered as its quantization.
iii) The special case of Theorem 6.2 where $\Theta = \{1,\ldots,n\}$ is equivalent to (3.5).

7. Characteristic polynomials and minimal polynomials

Suppose $A_{\Theta,\lambda}$ is as in (5.1). Because the *minimal polynomial* for a generic element of $A_{\Theta,\lambda}$ is $(t-\mu)(t-\nu)$, all entries of $(\mathbb{E}-\mu)(\mathbb{E}-\nu) \in M(n, S(\mathfrak{g}))$ vanish on $V_{A_{\Theta,\lambda}}$. Let $\widetilde{\mathcal{E}}$ be the linear subspace in $S(\mathfrak{g})$ spanned by these entries. It is G-stable by (3.2) and $\gamma^0(\widetilde{\mathcal{E}})$ is spanned by (4.5). Hence it follows from Lemma 4.1 that if $\mu \neq \nu$ then the system $\widetilde{\mathcal{E}}$ together with $\operatorname{Trace}\mathbb{E} - k\mu - (n-k)\nu$ generates $I^0_\Theta(\lambda) = \mathcal{I}^0_{\lambda_\Theta}$. A quantization of minimal polynomials can be formulated for general complex reductive Lie algebras:

Definition 7.1 ([**Os7**]). Suppose $\mathfrak{g}$ is a complex reductive Lie algebra and $\pi : \mathfrak{g} \to M(N,\mathbb{C})$ is its faithful representation. By the faithfulness we identify $\mathfrak{g}$ with $\pi(\mathfrak{g}) \subset M(N,\mathbb{C})$. Suppose moreover that the symmetric bilinear form $\langle X, Y\rangle = \operatorname{Trace} XY$ $(X, Y \in \pi(\mathfrak{g}))$ is nondegenerate on $\mathfrak{g}\times\mathfrak{g}$ (the assumption is automatic if $\mathfrak{g}$ is semisimple). Let $\pi^\vee$ denote the orthogonal projection of $M(N,\mathbb{C})$ onto $\mathfrak{g}$ with respect to $\langle\cdot,\cdot\rangle$. If we put $\mathbb{F}_\pi = \big(\pi^\vee(E_{ij})\big)$ then $\mathbb{F}_\pi$ can be regarded as an element of $M(N, U^\epsilon(\mathfrak{g}))$ $(\forall\epsilon \in \mathbb{C})$. We call a monic polynomial $q(t) \in U^\epsilon(\mathfrak{g})^G[t]$ the *characteristic polynomial* of $\mathbb{F}_\pi$ (or π) if it satisfies $q(\mathbb{F}_\pi) = 0$ with the lowest degree and denote it by $q^\epsilon_\pi(t)$. Also, let $\mathcal{M}$ be a $U^\epsilon(\mathfrak{g})$-module. We call a monic polynomial $q(t) \in \mathbb{C}[t]$ the *minimal polynomial* of the pair $(\pi,\mathcal{M})$ if it satisfies $q(\mathbb{F}_\pi)\,\mathcal{M} = 0$ (namely each entry of $q(\mathbb{F}_\pi)$ annihilates $\mathcal{M}$) with the lowest degree and denote it by $q_{\pi,\mathcal{M}}(t)$.

Remark 7.2. i) There always exists the characteristic polynomial $q^\epsilon_\pi(t)$. For any $\epsilon\in\mathbb{C}$, $U^\epsilon(\mathfrak{g})^G\simeq S(\mathfrak{a})^W$ by the Harish-Chandra isomorphism Γ^ϵ where $S(\mathfrak{a})^W$ is the algebra of Weyl group invariants in the symmetric algebra of a Cartan subalgebra $\mathfrak{a}$ (cf. (3.1)). The explicit formula of $q^\epsilon_\pi(t)$, regarded as an element of $S(\mathfrak{a})^W[t]$, is calculated by M. D. Gould [**Go2**]. In this formula we can interpret ϵ as an indeterminant because $q^\epsilon_\pi(t)\in S(\mathfrak{a})^W[t]$ polynomially depends on ϵ. We let $q_\pi(t)\in S(\mathfrak{a})^W[t,\epsilon]$ be such an interpretation of $q^\epsilon_\pi(t)$.
ii) $\operatorname{Trace}\mathbb{F}^m_\pi\in U^\epsilon(\mathfrak{g})^G$ $(m=0,1,\ldots)$ and $\Gamma^\epsilon\bigl(\operatorname{Trace}\mathbb{F}^m_\pi\bigr)$ is calculated by M. D. Gould [**Go1**].
iii) If a $U^\epsilon(\mathfrak{g})$-module $\mathcal{M}$ has a finite length or if it has an infinitesimal character, then there exists the minimal polynomial $q_{\pi,\mathcal{M}}(t)$.

For example, if $\mathfrak{g}=\mathfrak{o}_n$ and π is its natural representation then $\mathbb{F}_\pi=\Bigl(\frac{E_{ij}+E_{ji}}{2}\Bigr)$. If $\mathfrak{g}=\mathfrak{gl}_n$ and π is its natural representation then $\mathbb{F}_\pi=\mathbb{E}$ and for any increasing sequence Θ and $\lambda\in\mathbb{C}^{\#\Theta}$ the polynomial $q_{\pi,M^0_\Theta(\lambda)}(t)$ coincides with the minimal polynomial of a generic element of $A_{\Theta,\lambda}$ in the sense of linear algebra.

Theorem 7.3 ([**Os7**]). *Suppose $\mathfrak{g}=\mathfrak{gl}_n$ and π is its natural representation.*
i) *The characteristic polynomial of $\mathbb{E}$ is given by*

$$q_\pi(t)=\det\Bigl(-E_{ij}+\bigl(t-\epsilon(n-j)\bigr)\delta_{ij}\Bigr)=(-1)^n\times\Bigl(D(t)\ \textit{defined by}\ (3.4)\Bigr) \tag{7.1}$$

and it holds that $q_\pi(\mathbb{E})=0$ (a quantized Cayley-Hamilton *theorem).*
ii) *For any Θ and $\lambda\in\mathbb{C}^L$ $(L=\#\Theta)$, the minimal polynomial of $(\pi,M^\epsilon_\Theta(\lambda))$ is*

$$q_{\pi,M^\epsilon_\Theta(\lambda)}=\prod_{k=1}^{L}(t-\lambda_k-\epsilon n_{k-1}). \tag{7.2}$$

Let $\widetilde{\mathcal{E}}^\epsilon$ be the linear subspace of $U^\epsilon(\mathfrak{g})$ spanned by the n^2 entries of $q_{\pi,M^\epsilon_\Theta(\lambda)}(\mathbb{E})$. Then $\widetilde{\mathcal{E}}^\epsilon$ is G-stable and $\widetilde{\mathcal{E}}^\epsilon$ together with $\bigl\{Z_m-\gamma^\epsilon(Z_m)(\lambda_\Theta);\ m=1,\ldots,L-1\bigr\}$ generates $I^\epsilon_\Theta(\lambda)$ for a generic λ (it is sufficient if all $w.\lambda_\Theta$ $(w\in\mathfrak{S}_n)$ are distinct or if $\epsilon=0$ and all entries of $\lambda\in\mathbb{C}^L$ are distinct). Here $Z_m(m=1,\ldots,L-1)$ are the elements of $U^\epsilon(\mathfrak{g})^G$ defined by (3.3).

For a general $\mathfrak{g}$ we have the following:

Theorem 7.4. *Suppose $\mathfrak{g}$ and π are as in Definition 7.1. Let $M^\epsilon_\Theta(\lambda)$ be the scalar generalized Verma module for a standard parabolic subalgebra $\mathfrak{b}_\Theta$ and its character $\lambda\in(\mathfrak{b}_\Theta/[\mathfrak{b}_\Theta,\mathfrak{b}_\Theta])^*$ (the subscript Θ is a suitable parameter specifying the standard parabolic subalgebra). Then there exists a polynomial $q^\epsilon_{\pi,\Theta}(t;\lambda)$ in t,λ and ϵ such that $q_{\pi,M^\epsilon_\Theta(\lambda)}(t)=q^\epsilon_{\pi,\Theta}(t;\lambda)$ for a generic λ (the equality holds if all the roots of $q^\epsilon_{\pi,\Theta}(t;\lambda)$ as a polynomial in the single variable t are simple). Moreover, for any fixed $\epsilon\in\mathbb{C}$ and λ the divisibility relation $q_{\pi,M^\epsilon_\Theta(\lambda)}(t)\,|\,q^\epsilon_{\pi,\Theta}(t;\lambda)$ holds in $\mathbb{C}[t]$. We call $q^\epsilon_{\pi,\Theta}(t;\lambda)$ the* global minimal polynomial *of (π,Θ).*

Remark 7.5. i) The explicit form of $q^\epsilon_{\pi,\Theta}(t;\lambda)$ is determined by [**Os7**] in the case where $\mathfrak{g}$ is any classical Lie algebra and π is its natural representation. That in the fully general case is determined by [**OO**].
ii) If $\mathfrak{g}=\mathfrak{o}_n$ or $\mathfrak{sp}_n$ and π is its natural representation then the explicit form of $q_{\pi,M^\epsilon_\Theta(\lambda)}(t)$ for any λ is determined by [**Od3**]. (That for $\mathfrak{g}=\mathfrak{gl}_n$ is given by (7.2).)

iii) Let Θ_0 denote the Θ specifying the Borel subalgebra $\mathfrak{b}$. Then $q^{\epsilon}_{\pi,\Theta_0}(t;\lambda)$ ($\lambda \in \mathfrak{a}^* \simeq (\mathfrak{b}/[\mathfrak{b},\mathfrak{b}])^*$) equals the polynomial obtained by evaluating each coefficient of the characteristic polynomial $q_\pi(t) \in S(\mathfrak{a})^W[t,\epsilon]$ at $\lambda + \epsilon\rho$. Here $\epsilon\rho = \frac{1}{2}\operatorname{Trace}\bigl(\mathrm{ad}_{[\mathfrak{b},\mathfrak{b}]}\bigr)$.
iv) If π satisfies a certain additional condition then for any Θ and a generic λ we can construct a generating system of $I_\Theta(\lambda) = \operatorname{Ann}\bigl(M_\Theta(\lambda)\bigr)$ in the same way as Theorem 7.3 ii). (For instance, it is possible if $\mathfrak{g}$ is simple and π is the adjoint representation or a faithful representation with the lowest dimension.)

8. Integral geometry — Poisson transform and Penrose transform

In the case of $\mathfrak{g} = \mathfrak{gl}_n$, we have two different generating systems for the two-sided ideal $I^{\epsilon}_\Theta(\lambda)$ in $U^\epsilon(\mathfrak{g})$, which are respectively given by Theorem 6.2 and by Theorem 7.3 ii). (As for their relation, see [**Sak**].) The next theorem says the ideal $I^\epsilon_\Theta(\lambda)$ has the role of filling the *gap* between the two left ideals of $U^\epsilon(\mathfrak{g})$, $J^\epsilon_\Theta(\lambda)$ (the denominator of $M^\epsilon_\Theta(\lambda)$) and $J^\epsilon(\lambda_\Theta)$ (the denominator of $M^\epsilon(\lambda_\Theta)$). This property is important in application to integral geometry.

Theorem 8.1. *It holds for a generic λ that*

$$J^\epsilon_\Theta(\lambda) = I^\epsilon_\Theta(\lambda) + J^\epsilon(\lambda_\Theta) \qquad \textit{(GAP)}. \tag{8.1}$$

Remark 8.2. i) The theorem is valid for all complex reductive Lie algebras. In the case where $\epsilon = 1$, there is a sufficient condition for (8.1) given by Bernstein–Gelfand [**BG**] and A. Joseph [**Jos**], while [**OO**] obtains some conditions finer than it through the explicit calculations of $q^\epsilon_{\pi,\Theta}(t;\lambda)$ for various π. When $\epsilon = 0$, (8.1) holds if $I^0_\Theta(\lambda) = \mathcal{I}^0_{\lambda_\Theta}$.
ii) In the case of $\mathfrak{g} = \mathfrak{gl}_n$ a necessary and sufficient condition for (8.1) is given by [**Os4**]. For example, (8.1) is valid if $w.\lambda_\Theta$ ($w \in \mathfrak{S}_n$) are all distinct.

Hereafter, we assume that $\epsilon = 1$ and that $G_\mathbb{R}$ is a real form of $G = GL(n,\mathbb{C})$ or $G = SL(n,\mathbb{C})$ such as $GL(n,\mathbb{R}), U(p,q)$ and $SU^*(2m)$, or $G_\mathbb{R} = GL(n,\mathbb{C})$ as a real form of $G = GL(n,\mathbb{C}) \times GL(n,\mathbb{C})$. (More generally we may assume $G_\mathbb{R}$ is a real form of a connected complex reductive Lie group G.) Let P be a minimal parabolic subgroup of $G_\mathbb{R}$ and P_Ξ a parabolic subgroup containing P. Thus $G_\mathbb{R}/P_\Xi$ is a *generalized flag variety.* Let K be a maximal compact subgroup of $G_\mathbb{R}$ and λ a one-dimensional representation of P_Ξ such that $\lambda(P_\Xi \cap K) = \{1\}$. Put

$$\mathcal{B}(G_\mathbb{R}/P_\Xi;\lambda) := \bigl\{f \in \mathcal{B}(G_\mathbb{R});\ f(xp) = \lambda(p)f(x) \quad (\forall p \in P_\Xi)\bigr\}.$$

It is the space of hyperfunction sections of the line bundle on G/P_Ξ associated to λ^{-1}. When the action of $g \in G_\mathbb{R}$ on $\mathcal{B}(G_\mathbb{R}/P_\Xi;\lambda)$ is given by $L_g : f(x) \mapsto f(g^{-1}x)$, it is called a *degenerate principal series representation.* Since the Lie algebra $\mathfrak{g}$ of G is the complexification of the Lie algebra $\mathfrak{g}_\mathbb{R}$ of $G_\mathbb{R}$, the differential action $L_D \in \operatorname{End}\mathcal{B}(G_\mathbb{R}/P_\Xi;\lambda)$ is defined for any $D \in U(\mathfrak{g})$. Now we note that the complexification of the Lie algebra of P_Ξ equals $\mathfrak{b}_\Theta$ for some Θ and that the differential representation of λ is a character of $\mathfrak{b}_\Theta$ (also denoted by λ). It is not so hard to show the following equality holds:

$$\operatorname{Ann}\bigl(\mathcal{B}(G_\mathbb{R}/P_\Xi;\lambda)\bigr) := \bigl\{D \in U(\mathfrak{g});\ L_D f = 0\ (\forall f \in \mathcal{B}(G_\mathbb{R}/P_\Xi;\lambda))\bigr\} = {}^tI_\Theta(\lambda). \tag{8.2}$$

Here the rightmost side is the image of $I_\Theta(\lambda)$ under the antiautomorphism $D \mapsto {}^tD$ of $U(\mathfrak{g})$ induced by $\mathfrak{g} \ni X \mapsto -X \in \mathfrak{g}$. Therefore, for any given $G_\mathbb{R}$-map of

$\mathcal{B}(G_{\mathbb{R}}/P_{\Xi};\lambda)$ into the space of functions or line bundle sections on some other $G_{\mathbb{R}}$-homogeneous space, the image always satisfies the system of differential equations corresponding to ${}^tI_{\Theta}(\lambda)$. We remark such a $G_{\mathbb{R}}$-map is usually given by an integral operator since $G_{\mathbb{R}}/P_{\Xi}$ is compact.

EXAMPLE 8.3 (Grassmannians). Let $\mathbb{F}=\mathbb{R},\mathbb{C}$, or $\mathbb{H}$. The manifold $\mathrm{Gr}_k(\mathbb{F}^n)$ which consists of all k-dimensional linear subspace in $\mathbb{F}^n$ is called the *Grassmann manifold* and is an important example of generalized flag varieties. For example, if $\mathbb{F}=\mathbb{R}$,

$$\begin{aligned}\mathrm{Gr}_k(\mathbb{R}^n) &:= \{k\text{-dimensional linear subspace} \subset \mathbb{R}^n\} \quad \text{(real Grassmann manifold)}\\ &= M^\circ(n,k;\mathbb{R})/GL(k,\mathbb{R})\end{aligned}$$

where $M^\circ(n,k;\mathbb{R}) := \big\{X\in M(n,k;\mathbb{R});\ \operatorname{rank} X = k\big\}$. In addition, if we let $G_{\mathbb{R}} = GL(n,\mathbb{R})$ act on $\mathrm{Gr}_k(\mathbb{R}^n)$ by $g\circ X = {}^tg^{-1}X$, then we have

$$\mathrm{Gr}_k(\mathbb{R}^n) = GL(n,\mathbb{R})/P_{k,n} \simeq O(n)/O(k)\times O(n-k)$$

where

$$P_{k,n} := \left\{p=\begin{pmatrix} g_1 & 0\\ y & g_2\end{pmatrix};\ g_1\in GL(k,\mathbb{R}), g_2\in GL(n-k,\mathbb{R}), y\in M(n-k,k;\mathbb{R})\right\}.$$

Now we identify $\lambda=(\mu,\nu)\in\mathbb{C}^2$ with the character $p\mapsto|\det g_1|^\mu|\det g_2|^\nu$ of $P_{k,n}$ and consider

(8.3)

$$\begin{aligned}\mathcal{B}(G_{\mathbb{R}}/P_{k,n};\lambda) &= \big\{f\in\mathcal{B}(G_{\mathbb{R}});\ f(xp)=f(x)|\det g_1|^\mu|\det g_2|^\nu \quad (\forall p\in P_{k,n})\big\}\\ &\simeq \mathcal{B}(O(n)/O(k)\times O(n-k)).\end{aligned}$$

In this case, $\Theta=\{k,n\}$ and the ideal $I_\Theta(\lambda) = {}^t\operatorname{Ann}\big(\mathcal{B}(G_{\mathbb{R}}/P_{k,n};\lambda)\big)$ contains the determinant-type differential operators of order $k+1$ and $n-k+1$ given by (5.3), the second order differential operators of Theorem 7.3 ii), and the first order differential operator $Z_1-\gamma(Z_1)(\lambda_\Theta)$ coming from Trace. Notice that if $\nu=0$ then (8.3) is also isomorphic to

$$\big\{f\in\mathcal{B}(M^\circ(n,k;\mathbb{R}));\ f(Xg_1)=f(X)|\det g_1|^{-\mu}\quad(\forall g_1\in GL(k,\mathbb{R}))\big\}$$

as a $G_{\mathbb{R}}$-module.

Poisson transform.

We call the $G_{\mathbb{R}}$-map

$$\begin{array}{ccc}\mathcal{P}_{\Xi,\lambda}:\mathcal{B}(G_{\mathbb{R}}/P_{\Xi};\lambda) & \longrightarrow \big(\subset\mathcal{B}(G_{\mathbb{R}}/P;\lambda)\xrightarrow{\mathcal{P}_\lambda}\big) & \mathcal{A}(G_{\mathbb{R}}/K;\mathcal{M}_\lambda)\\ \Psi & & \Psi\\ f & \longmapsto & (\mathcal{P}_\lambda f)(x)=\displaystyle\int_K f(xk)dk\end{array}$$

a *Poisson transform*. Here, $\mathcal{M}_\lambda$ is a maximal ideal attached to λ in the algebra of invariant differential operators on the Riemannian symmetric space $G_{\mathbb{R}}/K$, and $\mathcal{A}(G_{\mathbb{R}}/K;\mathcal{M}_\lambda)$ is the solution space for it. We remark $G_{\mathbb{R}}/P_{\Xi}$ is isomorphic to a part of the *boundary* of a certain realization of $G_{\mathbb{R}}/K$.

Suppose $P_{\Xi}=P$ for a while. Then for a suitable λ the Poisson transform $\mathcal{P}_\lambda$ is a topological isomorphism of $\mathcal{B}(G_{\mathbb{R}}/P;\lambda)$ onto $\mathcal{A}(G_{\mathbb{R}}/K;\mathcal{M}_\lambda)$. This fact is observed by Helgason in some special cases. (For instance, if $G=SL(2,\mathbb{R})$ and $\lambda=0$ then a harmonic functions on the unit disk is the Poisson integral of a hyperfunction on the unit circle.) He gives in the general case a necessary and sufficient condition on λ for the injectivity of $\mathcal{P}_\lambda$ (cf. [**He1**]) and conjectures that the surjectivity also holds under the same condition. *Helgason's conjecture* is proved by [**K-**].

Now suppose P_Ξ is arbitrary and λ satisfies the condition for the bijectivity of $\mathcal{P}_\lambda$ (it is sufficient if λ is trivial). Let us concentrate on the problem of giving a concrete system of differential equations on $G_\mathbb{R}/K$ which characterizes the image of $\mathcal{P}_{\Xi,\lambda}$,

$$\mathcal{P}_{\Xi,\lambda}\big(\mathcal{B}(G_\mathbb{R}/P_\Xi;\lambda)\big) = \mathcal{P}_\lambda\big(\mathcal{B}(G_\mathbb{R}/P_\Xi;\lambda)\big) \subset \mathcal{A}(G_\mathbb{R}/K;\mathcal{M}_\lambda).$$

When λ is trivial, the problem is known as *Stein's problem* and there are many studies on it. In particular when $G_\mathbb{R}/K$ is a bounded symmetric domain and P_Ξ is its *Shilov boundary*, various systems of differential equations are constructed (cf. [**BV**], [**La**], [**KM**]). Also, K. D. Johnson [**Joh**] gives a unified method of constructing differential equations which applies to any $G_\mathbb{R}/K$ and P_Ξ when λ is trivial. But this method is not explicit enough to give the concrete form of differential operators. On the other hand, N. Shimeno [**Sh**] studies a generalized version of this problem for a bounded symmetric domain $G_\mathbb{R}/K$, certain types of P_Ξ, and any λ. The systems of differential equations given in these works are called *Hua systems* after L. K. Hua [**Hu**], the mathematician who first studied this problem.

We now return to the general setting and assume (8.1) holds. Then we have

$$\mathcal{B}(G_\mathbb{R}/P_\Xi;\lambda) = \big\{f \in \mathcal{B}(G_\mathbb{R}/P;\lambda);\ L_D f = 0 \quad (\forall D \in {}^tI_\Theta(\lambda))\big\}.$$

It follows that the image of $\mathcal{P}_{\Xi,\lambda}$ is characterized by ${}^tI_\Theta(\lambda)$ (and $\mathcal{M}_\lambda$). Hence by applying ${}^t\cdot$ to any of those generating systems of $I_\Theta(\lambda)$ constructed in the previous sections, we obtain a concrete system of differential equations characterizing the image of $\mathcal{P}_{\Xi,\lambda}$.

Remark 8.4. i) Since most of the known Hua systems coincide with systems coming from minimal polynomials in §7, we can treat them from such a unified point of view. For example, in the case of the Shilov boundary of a bounded symmetric domain, the system of differential equations has order 2 or 3 according as the domain is of tube type or not. We can explain the reason by the degree of minimal polynomials. In the case of the Shilov boundary of $SU(p,q)/S(U(p)\times U(q))$, the degree of the minimal polynomial is 2 if $p=q$ and 3 otherwise. But there always exists a second order system even if $p \neq q$ (cf. [**BV**]). We can clarify this phenomenon by decomposing the $G_\mathbb{R}$-stable generating system coming from the minimal polynomial into the sum of K-submodules (cf. [**OSh**]). Moreover, our approach enables us to determine at least which elements from $\mathcal{M}_\lambda$ we should add to the system.
ii) When $G_\mathbb{R}$ is a classical Lie group and P_Ξ is a maximal parabolic subalgebra, the generating system of (8.2) coming from the minimal polynomial of the natural representation has order ≤ 3. But this is not the case when $G_\mathbb{R}$ is of exceptional type, P_Ξ is maximal, and the minimal polynomial is that of a faithful representation with the lowest dimension (cf. [**OO**]).
iii) The correspondence of function classes under $\mathcal{P}_\lambda$ is studied by [**BOS**].

Penrose transform.

Let $\mathfrak{b}_\Theta \subset \mathfrak{g}$ be a parabolic subgroup and B_Θ the corresponding parabolic subgroup of the complex Lie group G. Let $\mathcal{O}_\lambda$ denote the sheaf of holomorphic sections of the line bundle on G/B_Θ associated to a one-dimensional holomorhic representation λ of B_Θ (or $\mathfrak{b}_\Theta$). With respect to the natural action of $U(\mathfrak{g})$, $\mathcal{O}_\lambda$ is annihilated by ${}^tI_\Theta(-\lambda)$. Hence, for any $G_\mathbb{R}$-orbit V in G/B_Θ the local cohomology $H^m_V(\mathcal{O}_\lambda)$ is a $G_\mathbb{R}$-module which is annihilated by ${}^tI_\Theta(-\lambda)$. Accordingly, if $\mathcal{T}_{\mathrm{Pen}}$ is a map of $H^m_V(\mathcal{O}_\lambda)$ into the space S of line bundle sections on a $G_\mathbb{R}$-homogeneous space such as the Riemannian symmetric space (a *Penrose transform*), then the image of $\mathcal{T}_{\mathrm{Pen}}$ satisfies the system of differential equations corresponding to ${}^tI_\Theta(-\lambda)$.

For example, suppose $G = GL(2n,\mathbb{C})$, $G_\mathbb{R} = U(n,n)$ and V is the closed orbit of G/B_Θ with $\Theta = \{k, 2n\}$ (thus G/B_Θ is the complex Grassmann manifold $\mathrm{Gr}_k(\mathbb{C}^{2n})$). In the additional setting such that S is the space of sections of a line bundle on the bounded symmetric domain $G_\mathbb{R}/K = U(n,n)/U(n)\times U(n)$, H. Sekiguchi [**Se**] examines a Penrose transform and in particular proves the image coincieds with the space of holomorphic solutions for the system of differential equations based on (6.1). This system can be expressed by some determinant-type differential operators with constant coefficients because (5.2) holds for the generalized Capelli elements in the generating system.

9. Integral geometry — Radon transform, hypergeometric functions

Radon transform.

In general, a $G_\mathbb{R}$-map between $\mathcal{B}(G_\mathbb{R}/P_\Xi;\,\lambda)$ and $\mathcal{B}(G_\mathbb{R}/P_{\Xi'};\,\lambda')$ is an integral transform. When it is the integration over a family of submanifolds in $G_\mathbb{R}/P_\Xi$, we call it a $G_\mathbb{R}$-map of Radon transform type. Suppose $0<k<\ell<n$. If we identify the *Radon transform* between real Grassmann manifolds

$$\begin{array}{cccc} \mathcal{R}^k_\ell: & \mathcal{B}(\mathrm{Gr}_k(\mathbb{R}^n)) & \to & \mathcal{B}(\mathrm{Gr}_\ell(\mathbb{R}^n)) \\ & \cup\!\!\!\!| & & \cup\!\!\!\!| \\ & \phi & \mapsto & (\mathcal{R}^k_\ell\phi)(x) = \int_{O(\ell)/O(k)\times O(\ell-k)} \phi(xy)dy \end{array}$$

with the linear map

$$\mathcal{R}^k_\ell : \mathcal{B}(G_\mathbb{R}/P_{k,n};\,(\ell,0)) \to \mathcal{B}(G_\mathbb{R}/P_{\ell,n};\,(k,0)),$$

then remarkably the latter is a $G_\mathbb{R}$-map. If $k+\ell<n$, then $\mathcal{R}^k_\ell$ is injective and its image is characterized by ${}^tI_{\{k,n\}}((\ell,0)) = \mathrm{Ann}\big(\mathcal{B}(G_\mathbb{R}/P_{k,n};(\ell,0))\big)$. More precisely

THEOREM 9.1 ([**Os3**]). *Suppose* $0<k<\ell<n$ *and* $k+\ell<n$. *Then* $\mathcal{R}^k_\ell$ *is a topological isomorphism of* $\mathcal{B}(\mathrm{Gr}_k(\mathbb{R}^n))$ *onto*

$$\left\{ \Phi(X)\in\mathcal{B}(M^\circ(n,\ell;\mathbb{R}));\ \begin{array}{l} \text{i) } \Phi(Xg) = \Phi(X)|\det g|^{-k} \quad (\forall g\in GL(\ell,\mathbb{R})\,), \\ \text{ii) } \det\Big(\dfrac{\partial}{\partial x_{i_\mu j_\nu}}\Big)_{\substack{1\le\mu\le k+1\\ 1\le\nu\le k+1}}\Phi(X) = 0 \\ \qquad (\,1\le i_1<\cdots<i_{k+1}\le n,\ 1\le j_1<\cdots<j_{k+1}\le\ell\,) \end{array} \right\}$$

where

$$M^\circ(n,\ell;\mathbb{R}) = \Big\{X = (x_{ij})_{\substack{1\le i\le n\\ 1\le j\le\ell}} \in M(n,\ell;\mathbb{R});\ \mathrm{rank}\,X = \ell\Big\}. \tag{9.1}$$

REMARK 9.2. i) A similar result for the complex Grassmann manifolds is obtained by T. Higuchi [**Hi**].
ii) Another characterization of the image is given by T. Kakehi [**Ka**]. The inverse map is studied in some works such [**Ka**], [**GR**].

Hypergeometric functions.

For general $G_{\mathbb{R}}$ and P_{Ξ}, we assume $\mathfrak{b}_{\Theta}$ is the complexification of the Lie algebra of P_{Ξ} as in §8. Thus the nilradical $\mathfrak{n}_{\Theta}$ of $\mathfrak{b}_{\Theta}$ is stable under the adjoint action of P_{Ξ} and $P_{\Xi} \ni p \mapsto \det \mathrm{Ad}_{\mathfrak{n}_{\Theta}}(p) \in \mathbb{C}^{\times}$ defines a one-dimensional representation of P_{Ξ}. It is known that for any $f \in \mathcal{B}(G_{\mathbb{R}}/P_{\Xi};\, \det \mathrm{Ad}_{\mathfrak{n}_{\Theta}}^{-1})$ the integral

$$\int_K f(xk)dk \qquad (x \in G_{\mathbb{R}})$$

does not depend on x.

Definition 9.3 ([**Os3**]). Retain the setting above. Let Q_j $(j = 1, 2)$ be closed subgroups of $G_{\mathbb{R}}$ such that each has an open orbit in $G_{\mathbb{R}}/P_{\Xi}$. Let λ, μ_j be one-dimensional representations of P_{Θ}, Q_j respectively. Suppose λ is trivial on $K \cap P_{\Xi}$ and ϕ_j $(j = 1, 2)$ are functions on $G_{\mathbb{R}}$ satisfying

$$\phi_1(q_1xp) = \mu_1(q_1)\lambda(p)\phi_1(x) \qquad (q_1 \in Q_1, p \in P_{\Xi}), \tag{9.2}$$

$$\phi_2(q_2xp) = \mu_2(q_2)\lambda^*(p)\phi_2(x) \qquad (q_2 \in Q_2, p \in P_{\Xi}, \lambda^* = \lambda^{-1} \det \mathrm{Ad}_{\mathfrak{n}_{\Theta}}^{-1}). \tag{9.3}$$

Then we call the function

$$\Phi_{\phi_1,\phi_2}(x) := \int_K \phi_1(xk)\phi_2(k)dk \left(= \int_K \phi_1(k)\phi_2(x^{-1}k)dk \right) \tag{9.4}$$

a *hypergeometric function.*

Remark 9.4. i) $\Phi_{\phi_1,\phi_2}(x)$ satisfies many differential equations. That is, the action of the Lie algebra of Q_1 on the left, the action of the Lie algebra of Q_2 on the right, and the action of $I_{\Theta}(\lambda)$ on the right (or equivalently, the action of ${}^tI_{\Theta}(\lambda)$ on the left). We call the system consisting of all these differential equations a *hypergeometric differential system.* In many cases or instances we can expect its solution space has finite dimension and is spanned by the hypergeometric functions (9.4).
ii) Suppose $Q_1 = Q_2 = K$ and μ_j (j=1,2) are trivial. Then $\Phi_{\phi_1,\phi_2}(x)$ is a *spherical function.* It is characterized by the hypergeometric differential system and (9.4) gives its integral representation. In some cases where P_{Ξ} is not a minimal parabolic subgroup, $\Phi_{\phi_1,\phi_2}(x)$ is written by using Lauricella's hypergeometric function F_D (cf. [**Kr**]).
iii) When $Q_1 = K$ and $Q_2 = N$, $\Phi_{\phi_1,\phi_2}(x)$ is a *Whittaker vector*, which is discussed in §10.
iv) The relative invariance under the action of every connected component of Q_i $(i = 1, 2)$ cannot be expressed in terms of the Lie algebra action. In order to fill this gap we sometimes append some additional conditions to the hypergeometric differential system. For example suppose $G_{\mathbb{R}} = GL(n, \mathbb{R})$, $Q_2 = P_{\ell,n}$ $(1 < \ell < n)$. Then Q_2 consists of 4 connected components, each containing one of the following:

$$m_1 := I_n,\ m_2 := \begin{pmatrix} -1 & 0 \\ 0 & I_{n-1} \end{pmatrix},\ m_3 := \begin{pmatrix} I_{n-1} & 0 \\ 0 & -1 \end{pmatrix},\ m_4 := m_2m_3. \tag{9.5}$$

If $\lambda = (\mu, 0)$ and if a solution Φ of the hypergeometric differential system satisfies

$$\Phi(xm_3) = \Phi(x), \tag{9.6}$$

then we can regard Φ as a function on $M^{\circ}(n, \ell; \mathbb{R})$ by letting $\Phi(X) = \Phi(x)$ for $X = {}^tx^{-1}\begin{pmatrix} E_\ell \\ 0 \end{pmatrix}$. Hence in such a case (for example in the next theorem) we can realize the hypergeometric differential system on $M^\circ(n,\ell;\mathbb{R})$ by forcing (9.6) on the solutions.

Now in the setting of Theorem 9.1 we suppose $P_\Xi = P_{k,n}$, $Q_2 = P_{\ell,n}$, $\lambda = (\ell,0)$ and $\mu_2 = (-k,0)$. Let ϕ_2 be the kernel functions of $\mathcal{R}^k_\ell$. Thus for any $\phi \in \mathcal{B}(G_\mathbb{R}/P_{k,n};(\ell,0))$

$$\mathcal{R}^k_\ell\phi(x) = \int_{O(n)} \phi(k)\phi_2(x^{-1}k)dk.$$

Theorem 9.1 immediately implies

THEOREM 9.5. *Let P_Ξ, Q_2, λ, μ_2 and ϕ_2 be as above. Let m_i be as in* (9.5) $(i = 1,2,3,4)$. *Suppose $H_\mathbb{R}$ is a connected closed subgroup of $G_\mathbb{R} = GL(n,\mathbb{R})$ with complexification H such that $(H \times GL(k,\mathbb{C}), \mathbb{C}^n \boxtimes \mathbb{C}^k)$ is a prehomogeneous vector space.* (*When $k = 1$ the assumption is satisfied by any prehomogeneous vector space (H,V) defined over $\mathbb{R}$ as long as $\dim V = n$.*) *Put $Q_1 = {}^tH_\mathbb{R}$ and let μ_1 be any character of Q_1. Then each solution Φ of the hypergeometric differential system that additionally satisfies*

$$\Phi(xm_i) = \Phi(x) \qquad (i = 1,2,3,4)$$

has a unique integral representation in the form of (9.4), *in which ϕ_1 is a relative invariant hyperfunction on $M^\circ(n,k;\mathbb{R})$ corresponding to the character $H_\mathbb{R} \times GL(k,\mathbb{R}) \ni (h,g) \mapsto \mu_1({}^th^{-1})|\det g|^\ell$. Here we used the natural identification $M(n,k;\mathbb{C}) \simeq \mathbb{C}^n \boxtimes \mathbb{C}^k$. Conversely, such a relative invariant ϕ_1 on $M(n,k;\mathbb{R})^\circ$, or on $M(n,k;\mathbb{R})$, gives a solution of the hypergeometric differential system.*

In the special case of Theorem 9.5 where $k = 1$, $H_\mathbb{R} = \mathbb{R}_{>0} \times \cdots \times \mathbb{R}_{>0}$ and $\mu_1(h_1^{-1},\ldots,h_n^{-1}) = \prod_{i=1}^n h_i^{\alpha_j}$ $(\sum_{i=1}^n \alpha_i = -\ell)$, the hypergeometric differential system on (9.1) is explicitly written as

(9.7)
$$\begin{cases} \displaystyle\sum_{j=1}^{\ell} x_{ij}\frac{\partial\Phi}{\partial x_{ij}} = \alpha_i\Phi & (\,1 \le i \le n\,) \qquad \cdots \text{ the left action of } H_\mathbb{R}, \\ \displaystyle\sum_{\nu=1}^{n} x_{\nu i}\frac{\partial\Phi}{\partial x_{\nu j}} = -\delta_{ij}\Phi & (\,1 \le i,j \le \ell\,) \qquad \cdots \text{ the right action of } \mathfrak{gl}_\ell, \\ \displaystyle\frac{\partial^2\Phi}{\partial x_{i_1j_1}\partial x_{i_2j_2}} = \frac{\partial^2\Phi}{\partial x_{i_2j_1}\partial x_{i_1j_2}} & \begin{pmatrix} 1 \le i_1 < i_2 \le n \\ 1 \le j_1 < j_2 \le \ell \end{pmatrix} \qquad \cdots \text{ Capelli type.} \end{cases}$$

Moreover, if $\epsilon_i = \pm$ for $i = 1,\ldots,n$ then the integral representation of the hypergeometric function corresponding to the distribution $\phi_1(x_1,\ldots,x_n) = \prod_{i=1}^n x_{\epsilon_i}^{\alpha_i} + \prod_{i=1}^n x_{-\epsilon_i}^{\alpha_i}$ is reduced to

$$\Phi(\alpha,x) = \int_{t_1^2+\cdots+t_\ell^2=1} \prod_{i=1}^n \Bigl(\sum_{\nu=1}^{\ell} t_\nu x_{i\nu}\Bigr)_{\epsilon_i}^{\alpha_i}\omega \qquad (\omega \text{ is the surface element}). \tag{9.8}$$

This function is known as *Aomoto-Gelfand's hypergeometric function* (cf. [**Ao**], [**GG**]).

REMARK 9.6. i) If $n = 4$ and $\ell = 2$ in the last example, then (9.8) essentially reduces to Gauss's hypergeometric function.

ii) If the number of the $H \times GL(k, \mathbb{C})$-orbits in

$$M^\circ(n, k; \mathbb{C}) := \{X \in M(n, k; \mathbb{C});\ \operatorname{rank} X = k\}$$

is finite in Theorem 9.5, it is proved by T. Tanisaki [**Ta2**, Proposition 4.5] that the hypergeometric differential system is *holonomic* and the dimension of the space of its local solutions is finite. This condition is satisfied if the prehomogeneous vector space $(H \times GL(k, \mathbb{C}), \mathbb{C}^n \boxtimes \mathbb{C}^k)$ has finite orbits and such prehomogeneous vector spaces are classified by T. Kimura and others [**KK**].
iii) There are a version of hypergeometric functions attached to Penrose transforms. They are studied by H. Sekiguchi [**Se**].

10. Whittaker vectors

Suppose $G_\mathbb{R} = KAN$ is an Iwasawa decomposition and χ is an one-dimensional representation of N. In this section we consider the realization of a $G_\mathbb{R}$-module $\mathcal{V}$ as a submodule of

$$\mathcal{B}(G_\mathbb{R}/N; \chi) := \{f \in \mathcal{B}(G_\mathbb{R});\ f(xn) = \chi^{-1}(n) f(x) \quad (\forall n \in N)\}.$$

When $G_\mathbb{R} = GL(n, \mathbb{R})$, we may assume

$$K = O(n),\ A = \Big\{\exp\Big(\sum_{i=1}^n x_i E_{ii}\Big);\ x_i \in \mathbb{R}\Big\},\ N = \Big\{\exp\Big(\sum_{i>j} x_{ij} E_{ij}\Big);\ x_{ij} \in \mathbb{R}\Big\}$$

and

$$\chi\Big(\exp\Big(\sum_{i>j} x_{ij} E_{ij}\Big);\ x_{ij} \in \mathbb{R}\Big) = e^{c_1 x_{21} + c_2 x_{32} + \cdots + c_{n-1} x_{nn-1}}$$

for some $c_j \in \mathbb{C}$ $(j = 1, \ldots, n-1)$. If $\mathcal{V} = \mathcal{B}(G_\mathbb{R}/P_\Xi; \lambda)$ is realized in $\mathcal{B}(G_\mathbb{R}/N; \chi)$ then a K-fixed vector u in the realization satisfies

$$\begin{cases} u(kxn) = \chi(n)^{-1} u(x) & (\forall k \in O(n),\ \forall n \in N), \\ L_D u = 0 & (\forall D \in {}^t I_\Theta(\lambda)). \end{cases} \tag{10.1}$$

We generally call a solution of the above system of equations a *Whittaker vector*. Owing to the Iwasawa decomposition a Whittaker vector u is determined by its restriction $v := u|_A$ to A. Since we know the concrete form of a generating system of ${}^t I_\Theta(\lambda)$, we can explicitly write down the equation system which v should satisfy.

Now suppose $\mathcal{V} = \mathcal{B}(G_\mathbb{R}/P_{k,n}; (\mu, \nu))$ $(2 \le 2k \le n)$, a degenerate principal series representation on the real Grassmann manifold $G_\mathbb{R}/P_{k,n}$. Then we can see from the explicit form of the system for v that the condition for the existence of nontrivial v is

$$c_i c_{i+1} = c_{i_1} c_{i_2} \cdots c_{i_{k+1}} = 0 \quad (1 \le i < n,\ 1 \le i_1 < i_2 < \ldots < i_{k+1} < n).$$

For example, if

$$\begin{cases} c_i = 0 & (i = 2, 4, \ldots, 2k, 2k+1, 2k+2, \ldots), \\ c_{2j-1} \ne 0 & (j = 1, \ldots, k), \end{cases}$$

then the system of differential equations for v is written as

$$\begin{cases} E_i v = \nu v \qquad (i = 2k+1, 2k+2, \dots, n), \\ (E_{2j-1} + E_{2j})v = (\mu + \nu - 2j + k + 1)v, \\ \Big(\big(\frac{E_{2j-1}-E_{2j}}{2}\big)^2 - \big(\frac{E_{2j-1}-E_{2j}}{2}\big) + c_{2j-1}^2 e^{2(x_{2j-1}-x_{2j})}\Big)v = \frac{\mu-\nu-k+1}{2}\big(\frac{\mu-\nu-k+1}{2} - 1\big)v, \\ \qquad\qquad \text{where } j = 1, \dots, k, \quad E_p = \frac{\partial}{\partial x_p} \ (p = 1, \dots, n). \end{cases}$$

From this we can deduce the multiplicity of the realization is 2^k, while the realization satisfying the *moderate growth condition* has multiplicity one. A Whittaker vector with moderate growth is thus unique up to a scalar multiple and is expressed by a *modified Bessel function of the second kind.*

Remark 10.1. Further studies on Whittaker vectors are given in [**Os6**].

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