Finite multiplicity theorems

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Abstract

We find upper and lower bounds of the multiplicities of irreducible admissible representations π of a semisimple Lie group G occurring in the induced representations $\mathrm{Ind}_H^G \tau$ from irreducible representations τ of a closed subgroup H. As corollaries, we establish geometric criteria for finiteness of the dimension of $\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \tau)$ (induction) and of $\mathrm{Hom}_H(\pi|_H, \tau)$ (restriction) by means of the real flag variety G/P, and criteria for uniform boundedness of these multiplicities by means of the complex flag variety.

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1 Introduction

Let G be a connected real semisimple Lie group with finite center, and H a closed (not necessarily, reductive) subgroup with at most finitely many connected components. We consider the following two geometric conditions:

(HP) There exists an open H-orbit on the real flag variety G/P.

(HB) There exists an open H_c-orbit on the complex flag variety G_c/B.

Here P is a minimal parabolic subgroup of G, B is a Borel subgroup of a complex Lie group G_c with the complexified Lie algebra $\mathfrak{g}_c = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, and

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H_c a complex subgroup with Lie algebra $\mathfrak{h}_c = \mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C}$, where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H, respectively. The condition (HB) is equivalent to that G_c/H_c is a *spherical variety* (i.e. B_c has an open orbit on G_c/H_c) when $G \supset H$ are defined algebraically.

An analogous notation $P_H \subset H$ and $B_H \subset H_c$ will be applied when H is reductive. In this case we can consider also the following two conditions:

(PP) There exists an open P_H-orbit on the real flag variety G/P.

(BB) There exists an open B_H-orbit on the complex flag variety G_c/B.

Clearly, these four conditions on the pair (G, H) do not depend on the choice of parabolics, coverings or connectedness of the groups, but are determined locally, namely, only by the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. An easy argument (see Lemma 4.2) shows that the following implications hold. Here we consider (PP) and (BB) when H is reductive:

$$\begin{array}{ccccc} & & \text{(HB)} & & \\ & \nearrow\!\!\!\nearrow & & \searrow\!\!\!\searrow & \\ \text{(BB)} & & & & \text{(HP)} \\ & \searrow\!\!\!\searrow & & \nearrow\!\!\!\nearrow & \\ & & \text{(PP)} & & \end{array}$$

None of the converse implications is true:

Example 1.1 ([13, Example 2.8.6]). Let (G, H) be a triple product pair $({}'G \times {}'G \times {}'G, \Delta {}'G)$ with ${}'G$ being a simple Lie group. Then (HP) holds iff ${}'G$ is compact or ${}'\mathfrak{g} \simeq \mathfrak{so}(n, 1)$, (HB) holds iff ${}'\mathfrak{g} \simeq \mathfrak{su}(2)$, $\mathfrak{sl}(2, \mathbb{R})$, or $\mathfrak{sl}(2, \mathbb{C})$; (PP) holds iff ${}'G$ is compact; (BB) never holds. Such a triple product arises naturally in the analysis of the tensor product of two representations (see [4] for ${}'G = SL(2, \mathbb{R})$; [7] for ${}'G = SO(n, 1)$, for instance).

It should be noted that the two conditions (HB) and (BB) depend only on the complexifications $(\mathfrak{g}_c, \mathfrak{h}_c)$. It is known by the work of Brion, Krämer, and Vinberg–Kimelfeld [5, 17, 18, 28] that the geometric condition (HB) characterizes the multiplicity-free property of irreducible (algebraic) *finite dimensional* representations π in the induced representation $\mathrm{Ind}_H^G \tau$ with $\dim \tau = 1$ (i.e. (G, H) is a *Gelfand pair*), and that the condition (BB) characterizes the multiplicity-free property of the restriction $\pi|_H$ with respect to $G \downarrow H$ (i.e. (G, H) is a *strong Gelfand pair*). An extensive research has been made in the decades in connection with algebraic group actions, invariant theory, and symplectic geometry among others (e.g. [27]), but mostly in the framework of algebraic (finite dimensional) representations or in the case of compact subgroups H.

These beautiful classic results may play a guiding principle in considering what a natural generalization would be for non-compact subgroups H (or for non-Riemannian homogeneous spaces G/H), however, only a complete change of machinery has enabled us to prove finite/bounded multiplicity results for admissible representations. Namely, in order to overcome analytic difficulties arising from non-compact subgroups H and from infinite dimensional representations, we employ the theory of a system of partial differential equations with regular singularities, for which micro-local analysis gives a canonical method. Thus we establish in this paper that the above four geometric conditions (HP), (HB), (PP), and (BB) characterize finiteness/boundedness of the multiplicities of the induction/restriction for admissible representations of real reductive groups, respectively (see Theorems A–D below).

For a precise statement of our results, let $\widehat{G}_{\mathrm{ad}}$ denote the set of equivalence classes of irreducible admissible representations of G (see Definition 2.1), and $\widehat{G}_{\mathrm{f}}$ that of irreducible finite dimensional representations of G. We write $c_{\mathfrak{g},K}(\pi, \operatorname{Ind}_H^G \tau)$ for the multiplicity of the underlying $(\mathfrak{g}, K)$-module π_K of $\pi \in \widehat{G}_{\mathrm{ad}}$ occurring in the space of sections of the G-homogeneous vector bundle over G/H associated to $\tau \in \widehat{H}_{\mathrm{f}}$ (the topology of $\operatorname{Ind}_H^G \tau$ is not the main issue here owing to analytic elliptic regularity).

Theorem A (finite multiplicity theorem for induction)**.**
1) *If* (HP) *holds, then* $c_{\mathfrak{g},K}(\pi, \operatorname{Ind}_H^G \tau) < \infty$ *for any* $\pi \in \widehat{G}_{\mathrm{ad}}$ *and any* $\tau \in \widehat{H}_{\mathrm{f}}$.
2) *Suppose that* G*,* H *and* τ *are defined algebraically over* $\mathbb{R}$*. If* (HP) *fails, then for any algebraic representation* τ *of* H *there exists* $\pi \in \widehat{G}_{\mathrm{ad}}$ *such that* $c_{\mathfrak{g},K}(\pi, \operatorname{Ind}_H^G \tau) = \infty$*.*

An upper bound formula of the multiplicities is presented in Theorem 2.4, which is strong enough to give a proof of uniformly bounded multiplicity results under stronger assumptions (Theorems B and D below), and thus plays a central role throughout the paper. The algebraic assumption in the second statement of Theorem A is crucial. A counterexample without the algebraic assumption is illustrated in Example 3.6.

Concerning the uniform boundedness of the multiplicities for the induced representation, we may consider the following three kinds of conditions:

$$\sup_{\tau\in\widehat{H}_{\mathrm{f}}}\sup_{\pi\in\widehat{G}_{\mathrm{ad}}}\frac{1}{\dim\tau}c_{\mathfrak{g},K}(\pi,\operatorname{Ind}_H^G\tau)<\infty. \tag{1.1}$$

$$\sup_{\substack{\tau\in\widehat{H}_{\mathrm{f}}\\ \dim\tau=1}}\sup_{\pi\in\widehat{G}_{\mathrm{ad}}}c_{\mathfrak{g},K}(\pi,\operatorname{Ind}_H^G\tau)<\infty. \tag{1.2}$$

$$\sup_{\pi\in\widehat{G}_{\mathrm{ad}}}c_{\mathfrak{g},K}(\pi,C^\infty(G/H))<\infty. \tag{1.3}$$

Clearly, (1.1) $\Rightarrow$ (1.2) $\Rightarrow$ (1.3).

Needless to say, $\widehat{G}_{\mathrm{ad}}$ and $\widehat{H}_{\mathrm{ad}}$ depend heavily on real forms (G,H) of (G_c,H_c). Surprisingly, the following theorem (and also Theorem D) guarantees that the uniform boundedness condition of the multiplicities is determined only by the complexified Lie algebras $(\mathfrak{g}_c,\mathfrak{h}_c)$.

Theorem B (uniformly bounded theorem of multiplicities for induction)**.**
1) *The condition* (HB) *implies* (1.1) (*hence,* (1.2) *and* (1.3)*, too*).
2) *Suppose* (G,H) *is defined algebraically over* $\mathbb{R}$*. Then the conditions* (HB), (1.1)*, and* (1.2)*, are all equivalent. Further, if* H *is reductive, then* (1.3) *is equivalent to these conditions, too.*

Remark 1.2. Theorem B is classically known for compact Lie group G even in a stronger form [5, 18], i.e. the upper bound (1.3) is one. In contrast to the compact case, the upper bound (1.3) is often greater than one if H is noncompact. For instance, if (G,H) is a semisimple symmetric pair $(SL(p+q,\mathbb{R}),SO_0(p,q))$, then the upper bound (1.3) is no less than $(p+q)!\,/\,p!\,q!$ in view of the contribution of the most continuous principal series representations for G/H (cf. [3, 25]).

Remark 1.3. It is known that the condition (HB) is equivalent to the commutativity of the ring of G-invariant differential operators. Further, if H is compact then the condition (HB) is equivalent to that the Riemannian manifold G/H is a weakly symmetric space in the sense of Selberg.

Example 1.4. 1) If (G,H) is a symmetric pair, then the condition (HB) (and therefore (HP)) is always fulfilled. In particular, the uniform bounded estimate (1.1) holds by Theorem B. This sharpens an earlier work of van den Ban [2]:

$$c_{\mathfrak{g},K}(\pi,\operatorname{Ind}_H^G\tau)<\infty\quad\text{for any }\pi\in\widehat{G}_{\mathrm{ad}}\text{ and }\tau\in\widehat{H}_{\mathrm{f}}. \tag{1.4}$$

In our context, (1.4) is derived from a weaker geometric condition (HP) by Theorem A.
2) If G_c/H_c is a spherical variety then any real form (G, H) satisfies (HB). There are some few non-symmetric spherical varieties G_c/H_c such as $(\mathfrak{g}_c, \mathfrak{h}_c) = (\mathfrak{sl}(2n+1, \mathbb{C}), \mathfrak{sp}(n, \mathbb{C}))$, $(\mathfrak{so}(2n+1, \mathbb{C}), \mathfrak{gl}(n, \mathbb{C}))$, and $(\mathfrak{so}(7, \mathbb{C}), \mathfrak{g}_2(\mathbb{C}))$, and they have been classified in [5, 18]. Some non-symmetric 'real spherical homogeneous spaces' G/H such as $SU(n, n+1)/Sp(n, \mathbb{R})$, $SU(2p+1, 2q)/Sp(p, q)$, $G_2(\mathbb{R})/SL(3, \mathbb{R})$, $G_2(\mathbb{R})/SU(2, 1)$, etc. admit discrete series representations (i.e. irreducible unitary representations that occur in closed subspaces of the L^2-spaces) and some others like $SL(2n+1, \mathbb{R})/Sp(n, \mathbb{R})$ do not (see [12, Part I]).
3) If we take H to be a maximal unipotent subgroup N, then (HP) holds by the open Bruhat cell. The condition (HB) is satisfied iff G is quasi-split. Our general formula (Theorem 2.4) applied to this special case gives an exact estimate of multiplicities of generalized Whittaker vectors for generic parameter in comparison with the Kostant–Lynch theory ([16, 20]; see Remark 2.5).

In Theorems A and B we have allowed H to be non-reductive and π to be infinite dimensional, but have confined τ to be finite dimensional. In Theorems C and D below, we treat the case where both π and τ are allowed to be infinite dimensional. Let denote by $\mathrm{Hom}_H(\ , \ )$ the space of continuous H-intertwining operators.

Theorem C (finite multiplicity theorem for restriction)**.** *Assume H is reductive in G.*
1) *If* (PP) *holds, then* $\dim \mathrm{Hom}_H(\pi|_H, \tau) < \infty$ *for any* $\pi \in \widehat{G}_{\mathrm{ad}}$ *and for any* $\tau \in \widehat{H}_{\mathrm{ad}}$.
2) *Suppose* (G, H) *is defined algebraically over* $\mathbb{R}$. *If* (PP) *fails, then there exist* $\pi \in \widehat{G}_{\mathrm{ad}}$ *and* $\tau \in \widehat{H}_{\mathrm{ad}}$ *such that* $\dim \mathrm{Hom}_H(\pi|_H, \tau) = \infty$.

Remark 1.5. If $H = K$ then its minimal parabolic subgroup P_H coincides with K itself and the assumption (PP) is automatically satisfied because $G = KP$. In this simplest case, any irreducible representation τ is *finite dimensional* and our argument for Theorem C 1) using hyperfunctions recovers so-called Casselman's subrepresentation theorem for which an algebraic proof using Jacquet functors [6] is also known. Our proof of Theorem C 1) is given in Section 2, which in this special case includes an analytic proof to an earlier work of Harish-Chandra [8] that every irreducible quasi-simple representation of G has finite K-multiplicities (cf. Section 2), for which an algebraic proof is also known (cf. [29, Chapter 3]).

Concerning uniform boundedness for the multiplicities of the restriction, we consider the following three kinds of conditions:

$$\sup_{\tau \in \widehat{H}_{\mathrm{ad}}} \sup_{\pi \in \widehat{G}_{\mathrm{ad}}} \dim \mathrm{Hom}_H(\pi|_H, \tau) < \infty. \tag{1.5}$$

$$\sup_{\tau \in \widehat{H}_{\infty}} \sup_{\pi \in \widehat{G}_{\infty}} \dim \mathrm{Hom}_H(\pi|_H, \tau) < \infty. \tag{1.6}$$

$$\sup_{\tau \in \widehat{H}_{\mathrm{f}}} \sup_{\pi \in \widehat{G}_{\mathrm{f}}} \dim \mathrm{Hom}_H(\pi|_H, \tau) < \infty. \tag{1.7}$$

Here $\widehat{G}_\infty$ ($\subset \widehat{G}_{\mathrm{ad}}$) denotes the set of equivalence classes of irreducible smooth admissible representations. Clearly, (1.5) $\Rightarrow$ (1.6) $\Rightarrow$ (1.7).

Theorem D (uniformly bounded theorem of multiplicities for restriction)**.** *Assume H is reductive.*
1) *The condition* (BB) *implies* (1.5) (*hence,* (1.6) *and* (1.7)*, too*)*.*
2) *Assume (G, H) is defined algebraically over $\mathbb{R}$. Then* (BB)*,* (1.5)*,* (1.6)*, and* (1.7) *are all equivalent.*

Example 1.6. 1) Owing to the classification [17], the condition (BB) is equivalent to that $(\mathfrak{g}_c, \mathfrak{h}_c)$ is the direct sum of some copies of $(\mathfrak{sl}_n(\mathbb{C}), \mathfrak{gl}_{n-1}(\mathbb{C}))$ $(\mathfrak{o}_n(\mathbb{C}), \mathfrak{o}_{n-1}(\mathbb{C}))$, and the trivial ones up to outer automorphisms. Therefore the real forms such as $(SL(n,\mathbb{R}), GL(n-1,\mathbb{R}))$, $(SU(p,q), U(p-1,q))$, $(O(p,q), O(p-1,q))$ are examples of the pair (G, H) satisfying (BB), and therefore (PP), too.
2) The symmetric pair $(G,H) = (SO(n,1), SO(k) \times SO(n-k,1))$ is an example of the pair that satisfies the condition (PP) but does not satisfy (BB) for $1 < k < n$. Likewise $(G,H) = (SU(n,1), S(U(k) \times U(n-k,1)))$ and $(Sp(n,1), Sp(k) \times Sp(n-k,1))$ satisfy (PP) but not (BB).

Recently, 'multiplicity-one theorems' have been proved in [26], asserting that the upper bound (1.6) equals one for certain real forms (G, H) satisfying the property (BB), which gives a finer result than Theorem D 1). However it should be noted that the uniform bound (1.6) can be greater than one for some other real forms (G, H) satisfying (BB). For instance, the upper bound (1.6) equals 2 if $(G,H) = (SL(2,\mathbb{R}), GL(1,\mathbb{R})_+)$. Our approach here is based on the theory of systems of partial differential equations with regular singularities is completely different from [1, 26] which is based on the Gelfand–Kazhdan criterion.

Our approach using hyperfunction boundary value maps naturally connects multiplicities with the geometry of the real flag variety. As one of applications of Theorem 2.4 we can obtain the following geometric result from infinite dimensional representation theory:

Corollary E. *For any closed subgroup H of G, the number of open H-orbits on G/P does not exceed the order of the little Weyl group $W(\mathfrak{a})$.*

We now outline the paper. In Section 2 we give a quick review on 'hyperfunction boundary maps' where no assumption such as K-finiteness is required, and prove a formula for the upper bound of the multiplicities in Theorem 2.4, which is a key step to prove the upper estimates in Theorems A to D. Conversely, the proof for a lower estimate of the multiplicities is based on a straightforward generalization of the construction of the Poisson transform for symmetric spaces. Theorem 3.1 is a stepping stone for the lower estimates in Theorems A and C. Uniform boundedness of multiplicities is discussed in Section 4 based on Theorem 2.4, combined with the Borel–Weil theorem for parabolic subgroups and a structural result on principal series representations. Thus we prove the first statement of Theorem B. The second statement of Theorems B and D reduces to the classical finite dimensional results. In Section 5 we discuss multiplicities for the restriction of irreducible representations, and complete the proof of Theorems C and D as an application of results in Sections 2 and 4.

2 An upper bound of the multiplicities

Let G be a connected real semisimple Lie group with finite center, and $\mathfrak{g}$ its Lie algebra. Let $Z(\mathfrak{g})$ be the center of the enveloping algebra $U(\mathfrak{g})$ of the complexified Lie algebra $\mathfrak{g}_c$. Then $Z(\mathfrak{g})$ is a polynomial ring of rank $\mathfrak{g}$ generators, and the Harish-Chandra isomorphism gives a parameterization of maximal ideals of $Z(\mathfrak{g})$:

$$\mathrm{Hom}_{\mathbb{C}\text{-alg}}(Z(\mathfrak{g}), \mathbb{C}) \simeq \mathfrak{j}_c^*/W(\mathfrak{j}), \quad \chi_\lambda \longleftrightarrow \lambda,$$

where $\mathfrak{j}$ is a Cartan subalgebra of $\mathfrak{g}$ and $W(\mathfrak{j})$ is the Weyl group for the root system for $(\mathfrak{g}_c, \mathfrak{j}_c)$.

Let π be a continuous representation of G on a complete locally convex vector space V. Define V^∞ to be the subspace of vectors $v \in V$ for which $g \mapsto \pi(g)v$ is a C^∞ map from G into V. Then V^∞ is a dense, G-invariant subspace of V. Let π^∞ denote the restriction of π to V^∞. Then π^∞ is a continuous representation on V^∞ endowed with a natural Fréchet topology, and is called a *smooth representation*. It has a property that $(V^\infty)^\infty = V^\infty$. Following Harish-Chandra we call π is *quasi-simple* if π^∞ restricts to scalar multiplication on $Z(\mathfrak{g})$.

We fix a maximal compact subgroup K of G. We recall (see [29, Chapters 3,11], for some further details):

Definition 2.1. A continuous representation (π, V) of G of finite length is called *admissible* if one of the following equivalent conditions are satisfied:

(2.1) π^∞ is $Z(\mathfrak{g})$ finite.

(2.2) $\dim \operatorname{Hom}_K(\delta, \pi) < \infty$ for any irreducible representation δ of K.

Then the space V_K consisting of K-finite vectors of V is contained in V^∞, and we write π_K for the underlying $(\mathfrak{g}, K)$-module defined on V_K. We denote by $\widehat{G}_{\mathrm{ad}}$ the set of equivalence classes of irreducible, admissible representations of G on complete locally convex topological vector spaces, and by $\widehat{G}_\infty$ that of smooth ones. Here two continuous representations (π_1, V_1) and (π_2, V_2) of G are defined to be equivalent if there exists a homeomorphic G-homomorphism $T : V_1 \to V_2$. Naturally we may regard $\widehat{G}_{\mathrm{ad}} \supset \widehat{G}_\infty \supset \widehat{G}_{\mathrm{f}}$.

Let H be a closed subgroup of G. A G-equivariant vector bundle $\mathcal{V}_\tau := G \times_H V_\tau$ over G/H associates a finite dimensional representation (τ, V_τ) of H. Then we have a representation π of G on the space of sections

$$\mathcal{F}(G/H;\tau) \simeq \{f \in \mathcal{F}(G, V_\tau) : f(gh) = \tau(h)^{-1} f(g) \text{ for } h \in H, g \in G\},$$

where $\mathcal{F} = \mathcal{A}$, C^∞, $\mathcal{D}'$ or $\mathcal{B}$ denote the sheaves of analytic, smooth, distributions, or hyperfunctions, respectively.

For each $\lambda \in \mathfrak{j}_c^*$, the Lie algebra $\mathfrak{g}$ acts on

$$\begin{aligned} \mathcal{F}(G/H;\tau)_\lambda \equiv& \mathcal{F}(G/H;\tau,\chi_\lambda) \\ :=& \{f \in \mathcal{F}(G/H;\tau) : d\pi(D)f = \chi_\lambda(D)f \quad \text{for any } D \in Z(\mathfrak{g})\}. \end{aligned} \tag{2.3}$$

Let $E(G/H;\tau)_\lambda$ be the subspace consisting of K-finite vectors, which is independent of $\mathcal{F}$ as far as $\dim \tau < \infty$ by analytic elliptic regularity [9, Theorem

3.4.4] because $d\pi(C_G - 2C_K)$ is an elliptic operator, where C_G is the Casimir element of $\mathfrak{g}$, and C_K is that for $\mathfrak{k}$ with the induced symmetric bilinear form from the restriction of the Killing form of $\mathfrak{g}$.

The significance of the geometric condition (HP) is summarized as follows:

Theorem 2.2. *If there exists an open H-orbit on G/P, then the $(\mathfrak{g}, K)$-module $E(G/H;\tau)_\lambda$ is of finite length for any finite dimensional representation τ of H and any $\lambda \in \mathfrak{j}_c^*$. In particular, $C^\infty(G/H;\tau)_\lambda$ is an admissible representation of G.*

The first statement of Theorem A follows from Theorem 2.2. The main goal of this section is to give a quantitative estimate of Theorem 2.2, namely, an upper bound for the multiplicities of irreducible subquotients in $E(G/H;\tau)_\lambda$ under the condition (HP) (see Theorem 2.4). In the course of its proof, we prove Theorem 2.2, too.

Let us fix some notation. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ the Cartan decomposition corresponding to K, and take a Cartan subalgebra $\mathfrak{j}$ of $\mathfrak{g}$ such that $\mathfrak{a} := \mathfrak{j} \cap \mathfrak{s}$ is a maximal abelian subspace in $\mathfrak{s}$. We put $\mathfrak{t} = \mathfrak{k} \cap \mathfrak{j}$. Let $\mathfrak{j}_c$, $\mathfrak{a}_c$ and $\mathfrak{t}_c$ be the complexifications of $\mathfrak{j}$, $\mathfrak{a}$ and $\mathfrak{t}$, and let denote by $\mathfrak{j}_c^*$, $\mathfrak{a}_c^*$ and $\mathfrak{t}_c^*$ the spaces of complex linear forms on them, respectively. By the Killing form of $\mathfrak{g}_c$ we identify $\mathfrak{a}_c^*$ and $\mathfrak{t}_c^*$ with subspaces of $\mathfrak{j}_c^*$. Let $\Sigma(\mathfrak{j})$, $\Sigma(\mathfrak{t})$ and $\Sigma(\mathfrak{a})$ be the set of the roots for the pairs $(\mathfrak{g}_c, \mathfrak{j}_c)$, $(\mathfrak{m}_c, \mathfrak{t}_c)$ and $(\mathfrak{g}, \mathfrak{a})$, respectively, and let $W(\mathfrak{j})$, $W(\mathfrak{t})$ and $W(\mathfrak{a})$ be the associated Weyl groups. Here $\mathfrak{m}_c$ is the centralizer of $\mathfrak{a}_c$ in $\mathfrak{k}_c$. We fix compatible positive systems $\Sigma(\mathfrak{t})^+$, $\Sigma(\mathfrak{j})^+$ and $\Sigma(\mathfrak{a})^+$, and let ρ denote half the sum of roots in $\Sigma(\mathfrak{j})^+$ and we put $\rho_\mathfrak{t} = \rho|_\mathfrak{t}$ and $\rho_\mathfrak{n} = \rho|_\mathfrak{a}$. Naturally we have $\Sigma(\mathfrak{t})^+ \subset \Sigma(\mathfrak{j})^+$. Put $A = \exp \mathfrak{a}$ and let M be the centralizer of $\mathfrak{a}$ in K, $L := MA$, and N the maximal nilpotent subgroup of G corresponding to $\Sigma(\mathfrak{a})^+$. Then $P = LN = MAN$ is a minimal parabolic subgroup. We denote by $\mathbb{C}_{\rho_\mathfrak{n}}$ the one dimensional representation of P given by $p \mapsto |\det(\mathrm{Ad}(p) : \mathfrak{n} \to \mathfrak{n})|^{\frac{1}{2}}$. Its differential representation equals $\rho_\mathfrak{n}$ when restricted to $\mathfrak{j}$.

Given $(\zeta, V_\zeta) \in \widehat{L}_\mathrm{f}$, we extend it to a representation of P with trivial action of N, and define another irreducible representation of P by

$$V_{\zeta,P} := V_\zeta \otimes \mathbb{C}_{\rho_\mathfrak{n}}. \tag{2.4}$$

Similarly, a $\bar{P}$-module $V_{\zeta,\bar{P}} := V_\zeta \otimes \mathbb{C}_{\rho_{\bar{\mathfrak{n}}}}$ is defined. Let $\mathcal{V}_{\zeta,P} := G \times_P V_{\zeta,P}$ be a G-equivariant vector bundle over G/P associated to the P-module $V_{\zeta,P}$, and we write $\mathcal{F}(U; V_{\zeta,P})$ for the space of $\mathcal{F} = \mathcal{A}$, $\mathcal{B}$, C^∞ or $\mathcal{D}'$-valued sections for

$\mathcal{V}_{\zeta,P}$ on an open set U of G/P. We write $I_P^G(\zeta)$ for the underlying $(\mathfrak{g}, K)$ - module of the normalized principal series representation $\mathcal{F}(G/P; V_{\zeta,P})$. Then the $Z(\mathfrak{g})$-infinitesimal character of $I_P^G(\zeta)$ equals $d\zeta + \rho_{\mathfrak{t}} \in \mathfrak{j}_c^*$ where $d\zeta$ denotes the highest weight of the finite dimensional representation ζ of the Lie algebra $\mathfrak{m} + \mathfrak{a}$ with respect to $\Sigma(\mathfrak{t})^+$.

For two continuous representations π and π', we write $\mathrm{Hom}_G(\pi, \pi')$ for the space of continuous G-homomorphisms. For two $(\mathfrak{g}, K)$-modules E, E', we set

$$c_{\mathfrak{g},K}(E, E') := \dim \mathrm{Hom}_{(\mathfrak{g},K)}(E, E').$$

By a little abuse of notation, we also write $c_{\mathfrak{g},K}(\pi, E')$ for $c_{\mathfrak{g},K}(\pi_K, E')$ if π_K is the underlying $(\mathfrak{g}, K)$-module of $\pi \in \widehat{G}_{\mathrm{ad}}$. We recall the following fundamental results on $(\mathfrak{g}, K)$-modules and their globalizations:

Lemma 2.3. 1) *For any two admissible representations π, π' on complete, locally convex vector space, we have*

$$\dim \mathrm{Hom}_G(\pi, \pi') \leq c_{\mathfrak{g},K}(\pi_K, \pi'_K) \tag{2.5}$$

2) *For any two admissible $(\mathfrak{g}, K)$-modules E, E', there exist admissible representations π, π' of G such that the equality holds in* (2.5) *with $\pi_K \simeq E$ and $\pi'_K \simeq E'$.*

The first statement is easy . For 2), the choice of such globalizations is not unique. For example, we can take π and π' to be smooth representations, by the Casselman–Wallach completion [29, Theorem 11.6.7].

For $\pi \in \widehat{G}_{\mathrm{ad}}$, $(\tau, V_\tau) \in \widehat{H}_{\mathrm{f}}$, and $(\zeta, V_\zeta) \in \widehat{L}_{\mathrm{f}} \simeq \widehat{P}_{\mathrm{f}} \simeq \widehat{\bar{P}}_{\mathrm{f}}$, we define

$$\begin{aligned} c_{\mathfrak{g},K}(\pi, \mathrm{Ind}_H^G \tau) &:= \dim \mathrm{Hom}_{(\mathfrak{g},K)}(\pi_K, \mathcal{F}(G/H; \tau)), \\ c_{H\cap\bar{P}}(V_\zeta, V_{\tau,\bar{P}}) &:= \dim \mathrm{Hom}_{H\cap\bar{P}}(V_\zeta, V_{\tau,\bar{P}}). \end{aligned} \tag{2.6}$$

Then, (2.6) is independent of $\mathcal{F}$ because the image of a $(\mathfrak{g}, K)$-homomorphism is contained in $\mathcal{A}(G/H; \tau)$ by analytic elliptic regularity.

We set

$$\begin{aligned} W_\lambda &:= \{w \in W(\mathfrak{j}) : w\lambda = \lambda\}, \\ W(\mathfrak{j}; \lambda) &:= \{v \in W(\mathfrak{j}) : (v\lambda)_{|\mathfrak{a}} = \lambda_{|\mathfrak{a}}\}. \end{aligned} \tag{2.7}$$

We say $\lambda \in \mathfrak{j}_c^*$ is *regular* if $W_\lambda = \{e\}$.

The first main result of this paper is:

Theorem 2.4. *Let H be a closed subgroup of G and suppose $H\bar{P}$ is open in G. Then for any finite dimensional representation τ of H and any $\pi \in \widehat{G}_{\mathrm{ad}}$, we have*

$$c_{\mathfrak{g},K}(\pi, \operatorname{Ind}_H^G \tau) \leq \#\left(W(\mathfrak{t})\backslash W(\mathfrak{j};\lambda)\right) \sum_{\zeta \in \widehat{L}_{\mathrm{f}}} c_{\mathfrak{g},K}(\pi, I_P^G(\zeta)) \cdot c_{H\cap\bar{P}}(V_\zeta, V_{\tau,\bar{P}}). \tag{2.8}$$

Here $\lambda \in \mathfrak{j}_c^*$ is the $Z(\mathfrak{g})$-infinitesimal character of π. If λ is regular, then we have

$$c_{\mathfrak{g},K}(\pi, \operatorname{Ind}_H^G \tau) \leq \sum_{\zeta \in \widehat{L}_{\mathrm{f}}} c_{\mathfrak{g},K}(\pi, I_P^G(\zeta)) \cdot c_{H\cap\bar{P}}(V_\zeta, V_{\tau,\bar{P}}).$$

Note that $c_{\mathfrak{g},K}(\pi, I_P^G(\zeta))$ is nonzero only for finitely many $\zeta \in \widehat{L}_{\mathrm{f}}$ for a fixed $\pi \in \widehat{G}_{\mathrm{ad}}$. Further, $c_{\mathfrak{g},K}(\pi, I_P^G(\zeta))$ is finite for any π and ζ (in fact, it is uniformly bounded, see Proposition 4.1). Hence if HgP is open for some $g \in G$, then $c_{\mathfrak{g},K}(\pi, \operatorname{Ind}_H^G \tau) < \infty$, which follows from Theorem 2.4 with $\bar{P}$ replaced by gPg^{-1}.

Remark 2.5. 1) If G is compact, then $G = P = \bar{P}$ and the equality holds in (2.8), which is the Frobenius reciprocity theorem.
2) If $H = N$ then the assumption in Theorem 2.4 is satisfied. In particular, if $\zeta_{|\mathfrak{a}} \in \mathfrak{a}_c^*$ is generic, (2.8) implies the following inequality:

$$c_{\mathfrak{g},K}(I_P^G(\zeta), \operatorname{Ind}_N^G \tau) \leq \#W(\mathfrak{a}) \cdot \dim \zeta. \tag{2.9}$$

In this special case, our estimate (2.8) is best possible. Indeed the equality holds in (2.9) as was proved by T. Lynch [20, Theorem 6.4].
3) If (G, H) is a semisimple symmetric pair, then the assumption in Theorem 2.4 is also satisfied.
4) There exists an open H-orbit in G/P if and only if the number of the H-orbits in G/P is finite ([21]). An analogous statement does not hold for general parabolic subgroups P.
5) It is plausible that the condition (HP) is equivalent to have a 'generalized Cartan decomposition' $G = KAH$ with some split abelian subgroup A of G as we raised in [13, Conjecture 3.5] when H is reductive in G. However, we do not need '$G = KAH$' type decomposition in this paper.

Our machinery for the proof of Theorem 2.4 is the theory of regular singularities of a system of partial differential equations [10, 22]. We regard the group manifold G as a symmetric space $(G \times G)/\Delta G$, and apply the construction of taking the boundary values of $\mathcal{B}(G;\chi_\lambda)$ to the hyperfunction valued principal series of $G\times G$. If a solution $f\in\mathcal{B}(G;\chi_\lambda)$ defined in (2.10) below is ideally analytic at a boundary point of G in the compactification $\widetilde{G}$ constructed in [23, §1] then f is expressed as a sum of convergent series. If f is $(K\times K)$-finite then f is automatically ideally analytic at any boundary point and this expression was studied earlier by Harish-Chandra, Casselman and Miličić among others. However, in our setting, we cannot assume that f is $(K\times K)$-finite. The advantage of our approach is that the boundary maps are well-defined inductively even locally as $(\mathfrak{g}+\mathfrak{g})$-homomorphisms, which enables us to capture the left $\mathfrak{g}$-module $\mathcal{B}(G;\chi_\lambda,\tau)$ as a filtered module under the assumption (HP). We review briefly some results of [23] in a way that we need.

We let $G\times G$ act on the space $\mathcal{B}(G)$ of hyperfunctions on G from the left and right:

$$(\pi_L(g)\circ\pi_R(g')f)(x)=f(g^{-1}xg')\quad\text{for } (g,g')\in G\times G \text{ and } f\in\mathcal{B}(G).$$

For $\chi_\lambda\in\mathrm{Hom}_{\mathbb{C}\text{-alg}}(Z(\mathfrak{g}),\mathbb{C})\simeq\mathfrak{j}_c^*/W(\mathfrak{j})$ we define

$$\mathcal{B}(G;\chi_\lambda):=\{f\in\mathcal{B}(G):d\pi_L(D)f=\chi_\lambda(D)f\text{ for any } D\in Z(\mathfrak{g})\}.\tag{2.10}$$

The boundary value maps are defined inductively as $\mathfrak{g}+\mathfrak{g}$-maps as follows: We set

$$\Xi:=\{(\lambda,\mu)\in\mathfrak{j}_c^*\times\mathfrak{a}_c^*:(w\lambda)|_{\mathfrak{a}}=\mu\quad\text{for some } w\in W(\mathfrak{j})\}.$$

For $(\lambda,\mu)\in\Xi$, we consider the following finite set of irreducible representations of the Levi subgroup $L=MA$ defined as

$$A(\lambda,\mu):=\{\zeta\in\widehat{L}_{\mathrm{f}}:d\zeta+\rho_{\mathfrak{t}}\in W(\mathfrak{j})\lambda,\, d\zeta|_{\mathfrak{a}}=\mu\}.$$

Further, corresponding to the 'logarithmic terms', we recall from [23, Proposition 2.8] the multiplicity function $N:\Xi\to\mathbb{N}$ with the properties

$$\begin{cases} N_{\lambda,\mu}\le\#\{w\in W(\mathfrak{j}):(w\lambda)_{|\mathfrak{a}}=\mu\}/\#W(\mathfrak{t}),\\ N_{\lambda,\mu}\le 1\quad\text{ if } \langle\lambda,\alpha\rangle\ne 0 \text{ for any } \alpha\in\Sigma(\mathfrak{j}),\\ N_{w\lambda,\mu}=N_{\lambda,\mu}\quad\text{for any } w\in W(\mathfrak{j}).\end{cases}\tag{2.11}$$

For $\lambda\in\mathfrak{j}_c^*$ we define a finite set

$$I_\lambda:=\{(\mu,i):(\lambda,\mu)\in\Xi,\ i=1,\dots,N_{\lambda,\mu}\}.$$

Clearly $I_\lambda = I_{w\lambda}$ for any $w \in W(\mathfrak{j})$. Fix $Y \in \mathfrak{a}$ such that $\alpha(Y) > 0$ for $\alpha \in \Sigma(\mathfrak{a})^+$ and that $\nu(Y) \neq \mu(Y)$ whenever $\nu \neq \mu$ with $(\lambda, \nu), (\lambda, \mu) \in \Xi$, and we give a lexicographical order $\prec$ on I_λ by $(\nu, j) \prec (\mu, i)$ if and only if $\mathrm{Re}(\mu - \nu)(Y) + \epsilon \, \mathrm{Im}(\mu - \nu)(Y) + \epsilon^2(i - j) > 0$ for $0 < \epsilon \ll 1$.

Let U be an open set in $(G \times G)/(P \times \bar{P})$. Then we have the boundary value maps

$$\beta^i_\mu : \ \mathcal{B}(G; \chi_\lambda)_{\mu,i} \ \to \bigoplus_{\zeta \in A(\lambda,\mu)} \mathcal{B}(U; V_{\zeta,P} \boxtimes V_{\zeta^*,\overline{P}})$$

for each $(\mu, i) \in I_\lambda$ on the subspace $\mathcal{B}(G; \chi_\lambda)_{\mu,i}$ defined inductively by

$$\mathcal{B}(G; \chi_\lambda)_{\mu,i} := \begin{cases} \mathcal{B}(G; \chi_\lambda) & \text{if } (\mu, i) \text{ is the smallest,} \\ \bigcap\limits_{\substack{(\nu,j) \in I_\lambda \\ (\nu,j) \prec (\mu,i)}} \mathrm{Ker}\, \beta^j_\nu & \text{otherwise.} \end{cases}$$

The subspaces $\mathcal{B}(G; \chi_\lambda)_{\mu,i}$ with the partial order $\prec$ induces a gradation of $\mathcal{B}(G; \chi_\lambda)$, and we write $\mathrm{gr}\, \mathcal{B}(G; \chi_\lambda)$ for the corresponding graded module. Then the collection $\bar{\beta} = \oplus_{(\mu,i) \in I_\lambda} \overline{\beta^i_\mu}$ of the induced map

$$\overline{\beta^i_\mu} : \mathcal{B}(G; \chi_\lambda)_{\mu,i} / \mathrm{Ker}\, \beta^i_\mu \to \bigoplus_{\zeta \in A(\lambda,\mu)} \mathcal{B}(U; V_{\zeta,P} \boxtimes V_{\zeta^*,\overline{P}})$$

gives a $\mathfrak{g} \oplus \mathfrak{g}$-homomorphism:

$$\bar{\beta} : \mathrm{gr}\, \mathcal{B}(G; \chi_\lambda) \to \bigoplus_{(\mu,i) \in I_\lambda} \bigoplus_{\zeta \in A(\lambda,\mu)} \mathcal{B}(U; V_{\zeta,P} \boxtimes V_{\zeta^*,\overline{P}}).$$

Moreover $\bar{\beta}$ respects the action of the subgroup of $G \times G$ that stabilizes U.

Assume that $H\bar{P}$ is open in G. We set $U := (G \times H\bar{P})/(P \times \bar{P})$. Then we have a $(\mathfrak{g} \times H)$-homomorphism

$$\bar{\beta} : \mathrm{gr}\, \mathcal{B}(G; \chi_\lambda)_{K \times 1} \to \bigoplus_{(\mu,i) \in I_\lambda} \bigoplus_{\zeta \in A(\lambda,\mu)} \mathcal{B}(U; V_{\zeta,P} \boxtimes V_{\zeta^*,\overline{P}}).$$

It is important to note that Holmgren's uniqueness principle for hyperfunctions holds, i.e. if $u \in \mathcal{B}(G; \chi_\lambda)$ satisfies $\beta^i_\mu(u) = 0$ for all $(\mu, i) \in I_\lambda$, then u vanishes on an open subset of G (see [23, §3]). Therefore $\bar{\beta}$ is injective since

$\mathcal{B}(G;\chi_\lambda)_{K\times 1}\subset\mathcal{A}(G)$ by analytic elliptic regularity. Passing to $1\times\Delta(H)$-fixed vectors in the $\mathfrak{g}\times H\times H$-map

$$\bar\beta\otimes\mathrm{id}:\mathrm{gr}\,\mathcal{B}(G;\chi_\lambda)_{K\times 1}\otimes V_\tau\to\bigoplus_{(\mu,i)\in I_\lambda}\bigoplus_{\zeta\in A(\lambda,\mu)}\mathcal{B}(U;V_{\zeta,P}\boxtimes V_{\zeta^*,\overline{P}})\otimes V_\tau,$$

we get an injective $\mathfrak{g}$-map

$$\bar\beta:(\mathrm{gr}\,\mathcal{B}(G;\chi_\lambda)_K\otimes V_\tau)^{\Delta(H)}\to\bigoplus_{(\mu,i)\in I_\lambda}\bigoplus_{\zeta\in A(\lambda,\mu)}\left(\mathcal{B}(U;V_{\zeta,P}\boxtimes V_{\zeta^*,\overline{P}})\otimes V_\tau\right)^{\Delta(H)}.$$

In light of the natural isomorphism

$$\left(\mathcal{B}(H\bar P/\bar P;V_{\zeta^*,\bar P})\otimes V_\tau\right)^{\Delta(H)}\simeq\left(V_{\zeta^*,\bar P}\otimes V_\tau\right)^{\Delta(H\cap\bar P)}\simeq\mathrm{Hom}_{H\cap\bar P}(V_\zeta,V_{\tau,\bar P}),$$

we have thus

$$\left(\mathcal{B}(U;V_{\zeta,P}\boxtimes V_{\zeta^*,\overline{P}})\otimes V_\tau\right)^{\Delta(H)}\simeq\mathcal{B}(G/P;V_{\zeta,P})\otimes\mathrm{Hom}_{H\cap\bar P}(V_\zeta,V_{\tau,\bar P}).$$

Hence we have obtained an injective $(\mathfrak{g},K)$-homomorphism

$$\bar\beta:\mathrm{gr}\,\mathcal{B}(G/H;\tau,\chi_\lambda)_K\to\bigoplus_{\zeta\in A(\lambda,\mu)}I_P^G(\zeta)\otimes\mathbb{C}^{N_{\lambda,\mu}}\otimes\mathrm{Hom}_{H\cap\bar P}(V_\zeta,V_{\tau,\bar P}).$$

Since the set of irreducible subquotients of the $(\mathfrak{g},K)$-module $\mathcal{B}(G/H;\tau)_\lambda$ is the same with that of the graded $(\mathfrak{g},K)$-module $\mathrm{gr}\,\mathcal{B}(G/H;\tau,\chi_\lambda)_K\simeq\mathrm{gr}\,E(G/H;\tau)_\lambda$. This completes the proof of Theorem 2.2.

Let $\pi\in\widehat{G}_{\mathrm{ad}}$, and λ be its infinitesimal character. Then

$$\begin{aligned}&\mathrm{Hom}_{\mathfrak{g},K}(\pi_K,\mathcal{B}(G/H;\tau))\\ \simeq\,&\mathrm{Hom}_{\mathfrak{g},K}\left(\pi_K,(\mathcal{B}(G)\otimes V_\tau)^{\Delta(H)}\right)\\ =\,&\mathrm{Hom}_{\mathfrak{g},K}\left(\pi_K,(\mathcal{B}(G;\chi_\lambda)\otimes V_\tau)^{\Delta(H)}\right)\\ \subset\,&\mathrm{Hom}_{\mathfrak{g},K}\left(\pi_K,(\mathrm{gr}\,\mathcal{B}(G;\chi_\lambda)_K\otimes V_\tau)^{\Delta(H)}\right)\\ \subset\,&\bigoplus_{\zeta\in A(\lambda,\mu)}\mathbb{C}^{N_{\lambda,\mu}}\otimes\mathrm{Hom}_{\mathfrak{g},K}\left(\pi_K,I_P^G(\zeta)\right)\otimes\mathrm{Hom}_{H\cap\bar P}(V_\zeta,V_{\tau,\bar P})\end{aligned}$$

and hence

$$\begin{aligned}c_{\mathfrak{g},K}(\pi,\mathrm{Ind}_H^G\tau)&\le\sum_{\zeta\in A(\lambda,\mu)}N_{\lambda,\mu}c_{\mathfrak{g},K}(\pi_K,I_P^G(\zeta))\cdot c_{H\cap\bar P}(V_\zeta,V_{\tau,\bar P})\\&=\sum_{\zeta\in\widehat{L}_{\mathrm{f}}}N_{\lambda,d\zeta_{|\mathfrak{a}}}\cdot c_{\mathfrak{g},K}(\pi_K,I_P^G(\zeta))\cdot c_{H\cap\bar P}(V_\zeta,V_{\tau,\bar P}).\end{aligned}$$

Now Theorem 2.4 follows from (2.11). □

As we have seen in the proof of Theorem 2.4, the above argument leads us to an upper estimate of the multiplicities of subquotients as well. Let denote by $[E : \pi]$ the multiplicity of an irreducible $(\mathfrak{g}, K)$-module π_K occurring as a subquotient of a $(\mathfrak{g}, K)$-module E.

Proposition 2.6. *Suppose that $H\bar{P}$ is open. For any $\tau \in \widehat{H}_{\mathrm{f}}$ and any $\pi \in \widehat{G}$ having $Z(\mathfrak{g})$-infinitesimal character $\lambda \in \mathfrak{j}_c^*$, we have*

$$[E(G/H;\tau)_\lambda : \pi] \leq \#\left(W(\mathfrak{t})\backslash W(\mathfrak{j};\lambda)\right) \sum_{\zeta \in \widehat{L}_{\mathrm{f}}} [I_P^G(\zeta) : \pi] \cdot c_{H\cap\bar{P}}(V_{\zeta,\bar{P}}, V_\tau).$$

Corollary 2.7. *Suppose $H\bar{P}$ is open and $\mu \in \mathfrak{a}_c^*$ satisfies $\mathrm{Re}\langle\mu, \alpha\rangle \geq 0$ for any $\alpha \in \Sigma(\mathfrak{a})^+$. Assume that $\mu + \rho_{\mathfrak{t}} \in \mathfrak{j}_c^*$ is regular with respect to $W(\mathfrak{j})$. Then for any $\tau \in \widehat{H}_{\mathrm{f}}$ we have*

$$c_{\mathfrak{g},K}(I_P^G(\mathbf{1}\otimes\mu), C^\infty(G/H;\tau)) \leq \#W(\mathfrak{a}) \cdot c_{H\cap\bar{P}}(V_{\mathbf{1}\otimes\mu}, V_{\tau,\bar{P}}). \tag{2.12}$$

Proof of Corollary 2.7. Let π_K be the unique irreducible quotient of the spherical principal series representation $I_P^G(\mathbf{1}\otimes\mu)$. Since the K-fixed vector in $I_P^G(\mathbf{1}\otimes\mu)$ is cyclic (cf. [15]), we have

$$c_{\mathfrak{g},K}(I_P^G(\mathbf{1}\otimes\mu), C^\infty(G/H,\tau)) \leq c_{\mathfrak{g},K}(\pi_K, C^\infty(G/H;\tau)).$$

It follows from the theory of zonal spherical functions that $c_{\mathfrak{g},K}(\pi_K, I_P^G(\zeta)) \neq 0$ (or, equivalently, $= 1$) only if ζ is of the form $\mathbf{1}\otimes w\mu$ for some $w \in W(\mathfrak{a})$. Hence Corollary follows from (2.11) and from the last formula in the proof of Theorem 2.4. □

Example 2.8. μ satisfies the regularity condition of Corollary 2.7, in the following cases:
1) $\mu = \rho_{\mathfrak{n}}$.
2) $\mathrm{Im}\,\mu$ is regular with respect to $W(\mathfrak{a})$.

The case 1) is clear. Let us see the case 2). If $w \in W(\mathfrak{j})$ satisfies $w(\rho_{\mathfrak{t}} + \mu) = \rho_{\mathfrak{t}} + \mu$, then we have $w\,\mathrm{Im}\,\mu = \mathrm{Im}\,\mu$ by taking the projection to $\mathbb{R}$-span $\sqrt{-1}\,\Sigma(\mathfrak{j})$. By Chevalley's theorem, w is contained in the subgroup generated by the reflection of the roots orthogonal to $\mathrm{Im}\,\mu$, that is, $w \in W(\mathfrak{t})$ by the assumption. Now we have $\rho_{\mathfrak{t}} = (w(\rho_{\mathfrak{t}} + \mu))_{|\mathfrak{t}} = w\rho_{\mathfrak{t}}$, showing $w = 1$.

3 A lower bound of the multiplicities

In this section we give a proof of Theorem A 2) and Corollary E. The key idea is to generalize the construction of the Poisson transform known for symmetric spaces, see Theorem 3.1 below.

Let us recall how irreducible finite dimensional representations are realized into principal series representations. As before, let $P = LN$ be a Langlands decomposition of the minimal parabolic subgroup P of G, and $\mathfrak{n}$ the Lie algebra of N. Suppose σ is an irreducible finite dimensional representation of G on a vector space V_σ. Then, the Levi subgroup L leaves

$$V_\sigma^{\mathfrak{n}} := \{v \in V_\sigma : d\sigma(X)v = 0 \ \text{ for any } X \in \mathfrak{n}\}$$

invariant, and acts irreducibly on it. We denote by ζ_σ this representation of L. Then σ is the unique quotient of the principal series representation $I_P^G(\zeta_\sigma)$, or equivalently, the contragredient representation σ^* satisfies:

$$\dim \operatorname{Hom}_{\mathfrak{g},K}(\sigma^*, I_P^G(\zeta_\sigma^*)) = 1. \tag{3.1}$$

For $\sigma \in \widehat{G}_{\mathrm{f}}$ and $\tau \in \widehat{H}_{\mathrm{f}}$, we set

$$c_H(\sigma, \tau) := \dim \operatorname{Hom}_H(\sigma|_H, \tau).$$

The following lower bound of the dimension of $(\mathfrak{g}, K)$-homomorphisms is crucial in the proof of Theorem A 2) and Corollary E.

Theorem 3.1. *Suppose that H is a closed subgroup of G and that there are m disjoint H-invariant open sets of G/P. Then*

$$c_{\mathfrak{g},K}(I_P^G(\zeta_\sigma), \operatorname{Ind}_H^G \tau) \geq m c_H(\sigma, \tau)$$

for any $\sigma \in \widehat{G}_{\mathrm{f}}$ and $\tau \in \widehat{H}_{\mathrm{f}}$.

In order to prove Theorem 3.1, we construct $(\mathfrak{g}, K)$-homomorphisms from a principal series representation $I_P^G(\zeta)$ into $\operatorname{Ind}_H^G \tau$ by means of kernel hyperfunctions:

Lemma 3.2. *For any $\zeta \in \widehat{L}_{\mathrm{f}}$ and $(\tau, V_\tau) \in \widehat{H}_{\mathrm{f}}$, we have*

$$c_{\mathfrak{g},K}(I_P^G(\zeta), \operatorname{Ind}_H^G \tau) \geq \dim \left(V_\tau \otimes \mathcal{B}(G/P; V_{\zeta^*,P})\right)^H.$$

Here $\left(V_\tau \otimes \mathcal{B}(G/P; V_{\zeta^*,P})\right)^H$ denotes the space of H-fixed vectors of the diagonal action.

Proof. The natural G-invariant paring

$$\langle\ ,\ \rangle : \mathcal{A}(G/P; V_{\zeta,P}) \times \mathcal{B}(G/P; V_{\zeta^*,P}) \to \mathbb{C}$$

induces an injective G-homomorphism

$$\Psi : \mathcal{B}(G/P; V_{\zeta^*,P}) \hookrightarrow \operatorname{Hom}_G(\mathcal{A}(G/P, V_{\zeta,P}), \mathcal{A}(G))$$

by $\Psi(\chi)(u)(g) := \langle \pi(g^{-1})u, \chi\rangle$ for $\chi \in \mathcal{B}(G/P; V_{\zeta^*,P})$, $u \in \mathcal{A}(G/P, V_{\zeta,P})$ and $g \in G$. Here, we let G act on $\operatorname{Hom}_G(\mathcal{A}(G/P, V_{\zeta,P}), \mathcal{A}(G))$ via the right translation on $\mathcal{A}(G)$. Passing to the space of $\Delta(H)$-fixed vectors in the $G \times H$-map

$$\Psi \otimes \mathrm{id} : \mathcal{B}(G/P; V_{\zeta^*,P}) \otimes V_\tau \hookrightarrow \operatorname{Hom}_{\mathfrak{g},K}(\mathcal{A}(G/P; V_{\zeta,P}), \mathcal{A}(G)) \otimes V_\tau,$$

we have an injective map

$$\mathcal{P} : (V_\tau \otimes \mathcal{B}(G/P; V_{\zeta^*,P}))^H \hookrightarrow \operatorname{Hom}_G(\mathcal{A}(G/P; V_{\zeta,P}), \operatorname{Ind}_H^G \tau). \tag{3.2}$$

Hence we have proved Lemma 3.2. □

Example 3.3. If $H = K$ and τ is the one dimensional trivial representation, then $(V_\tau \otimes \mathcal{B}(G/P; V_{\zeta^*,P}))^K \neq 0$ (or, equivalently is one dimensional) if and only if $\zeta|_M$ is trivial. The corresponding intertwining operator (see (3.2)) from $\mathcal{A}(G/P; V_{\zeta^*,P})$ into $\mathcal{A}(G/K)$ coincides with the Poisson transform for the Riemannian symmetric space G/K up to a scalar multiple.

Proof of Theorem 3.1. Let U_i $(i = 1, 2, \ldots, m)$ be disjoint H-invariant open subsets of G/P. We define $\chi_i \in \mathcal{B}(G/P)$ by

$$\chi_i(g) = \begin{cases} 1 & \text{if } g \in U_i, \\ 0 & \text{if } g \notin U_i. \end{cases}$$

Clearly, $\chi_i \in \mathcal{B}(G/P)^H$ $(i = 1, 2, \ldots, m)$ are linearly independent.

Next we identify V_σ^* with the unique subspace of the principal series representation $I_P^G(\zeta_\sigma^*)$ (see (3.1)). Take linearly independent H-fixed elements $u_1, \ldots, u_n$ of $V_\tau \otimes V_\sigma^*$ with $n := c_H(\sigma, \tau)$, where we have regarded as $u_j \in \left(V_\tau \otimes \mathcal{B}(G/P; V_{\zeta_\sigma^*,P})\right)^H$. Then $\chi_i u_j \in \left(V_\tau \otimes \mathcal{B}(G/P; V_{\zeta_\sigma^*,P})\right)^H$ are well-defined and linearly independent for $i = 1, \ldots, m$ and $j = 1, \ldots, n$ because u_j are real analytic. Owing to Lemma 3.2, Theorem 3.1 has been now proved. □

We pin down special cases of Theorem 3.1:

Example 3.4. Suppose H is a closed subgroup of G.

1) For any $\sigma \in \widehat{G}_{\mathrm{f}}$ and $\tau \in \widehat{H}_{\mathrm{f}}$, $c_{\mathfrak{g},K}(I_P^G(\zeta_\sigma), \operatorname{Ind}_H^G \tau) \geq c_H(\sigma, \tau)$.

2) $c_{\mathfrak{g},K}(I_P^G(\mathbf{1}), C^\infty(G/H)) \geq m$.

The first statement follows because G/P itself is an open H-invariant subset, and the second statement corresponds to $\sigma = \mathbf{1}$, $\tau = \mathbf{1}$ and $\zeta_\sigma = \mathbf{1}$.

Finally, we use the following elementary result for algebraic groups.

Lemma 3.5. *Suppose H is an algebraic subgroup of a real algebraic semisimple Lie group G. If there is no open H-orbit on G/P, then there exist infinitely many, disjoint H-invariant open sets of G/P.*

For the sake of completeness, we give a proof of Lemma 3.5 in Appendix.

Proof of Theorem A 2). Suppose there is no open H-orbit on G/P. Then we can take infinitely many disjoint H-invariant open subsets U_i of G/P by Lemma 3.5. For a given algebraic representation $\tau \in \widehat{H}_{\mathrm{f}}$, we can take $\sigma \in \widehat{G}_{\mathrm{f}}$ with $c_H(\sigma, \tau) > 0$ by the Frobenius reciprocity. Then $c_{\mathfrak{g},K}(I_P^G(\zeta_\sigma), \operatorname{Ind}_H^G \tau) = \infty$ by Theorem 3.1. Since there are at most finitely many irreducible $(\mathfrak{g}, K)$-modules occurring in the principal series representation $I_P^G(\zeta_\sigma)$ as subquotients, Theorem 3.1 now follows. $\square$

Proof of Corollary E. Let m be the number of open H-orbits on G/P. By Example 3.4, we have

$$c_{\mathfrak{g},K}(I_P^G(\mathbf{1}), \operatorname{Ind}_H^G \mathbf{1}) \geq m.$$

Comparing this with Corollary 2.7 in the case $\mu = \rho_{\mathfrak{n}}$ and $\tau = \mathbf{1}$, we get $m \leq \# W(\mathfrak{a})$. $\square$

We end this section with a counterexample to an analogous multiplicity-finite statement without algebraic assumptions in Theorem 3.1.

Example 3.6. Let $G = SL(2, \mathbb{R}) \times \cdots \times SL(2, \mathbb{R})$ be the direct product group of $(n+1)$-copies of $SL(2, \mathbb{R})$. Fix real numbers $\lambda_1, \ldots, \lambda_n$ which are linearly independent over $\mathbb{Q}$. Writing $k_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and $p_{t,x} := \begin{pmatrix} e^t & x \\ 0 & e^{-t} \end{pmatrix}$, we define a two-dimensional subgroup of G by

$$H = \{g_{t,x} = (p_{t,x}, k_{\lambda_1 t}, \cdots, k_{\lambda_n t}) : (t,x) \in \mathbb{R}^2\}.$$

Then there is no open H-orbit on G/P if $n > 1$ because $\dim G/P = n+1 > \dim H = 2$. However, we still have a finite multiplicity statement:

$$c_{\mathfrak{g},K}(\pi, \operatorname{Ind}_H^G \tau) \leq 2 \quad \text{for any } \pi \in \widehat{G}_{\mathrm{ad}} \text{ and for any } \tau \in \widehat{H}_{\mathrm{f}}. \tag{3.3}$$

Let us prove (3.3). We observe that any finite dimensional irreducible representation of H factors through the quotient group $H/[H,H] \simeq \mathbb{R}$, and is of the form $\tau_\mu(g_{t,x}) = e^{\mu t}$ for some $\mu \in \mathbb{C}$. Let $\chi_m(k_\theta) := e^{2\pi\sqrt{-1}m\theta}$ and $\sigma_\mu(p_{t,x}) := e^{\mu t}$. Then χ_m ($m \in \mathbb{Z}$) and σ_μ ($\mu \in \mathbb{C}$) are one dimensional representations of $SO(2)$ and $AN = \{p_{t,x} : t, x \in \mathbb{R}\}$, respectively.

For $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ and $u \in C^\infty(G/H; \tau_\mu)$, we define

$$(S_m u)(g_0, g_1, \ldots, g_m) := \int_0^{2\pi} \cdots \int_0^{2\pi} u(g_0, g_1 k_{\theta_1}, \ldots, g_n k_{\theta_n}) e^{2\pi\sqrt{-1}(m_1\theta_1 + \cdots + m_n\theta_n)} \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_n}{2\pi}.$$

Then, for $t, x, \varphi_1, \ldots, \varphi_n \in \mathbb{R}/2\pi\mathbb{Z}$, and $g = (g_0, g_1, \cdots, g_n)$, we have

$$(S_m u)(g g_{t,x}) = \sigma_{\mu - 2\pi\sqrt{-1}(\lambda_1 m_1 + \cdots + \lambda_n m_n)}(p_{t,x}^{-1}) \prod_{j=1}^n \chi_{m_j}(k_{\varphi_j}^{-1})(S_m u)(g).$$

Thus, $S_m u$ defines an element of $C^\infty(G/\widetilde{H}; \sigma_{\mu - 2\pi\sqrt{-1}\langle\lambda,m\rangle} \otimes \chi_m)$ where $\chi_m = \chi_{m_1} \otimes \cdots \otimes \chi_{m_n}$, $\langle \lambda, m \rangle := \lambda_1 m_1 + \cdots + \lambda_n m_n$, and

$$\widetilde{H} := AN \times SO(2) \times \cdots \times SO(2).$$

Clearly, $S := \bigoplus_{m \in \mathbb{Z}^n} S_m$ gives an injective G-homomorphism:

$$S : C^\infty(G/H; \tau_\mu) \to \bigoplus_{m \in \mathbb{Z}^n} C^\infty(G/\widetilde{H}; \sigma_{\mu - 2\pi\sqrt{-1}\langle\lambda,m\rangle} \otimes \chi_m).$$

Now (3.3) follows from the well-known facts on $G_1 = SL(2,\mathbb{R})$:

1) $\#\{\mu \in \mathbb{C} : \operatorname{Hom}_{(\mathfrak{g}_1, K_1)}(\pi_1, \operatorname{Ind}_{AN}^{G_1} \sigma_\mu) \neq 0\} \leq 2$, for any $\pi_1 \in \widehat{G_1}_{\mathrm{ad}}$.

2) $\operatorname{Ind}_{K_1}^{G_1} \chi_l$ is multiplicity-free for any $l \in \mathbb{Z}$.

4 Uniform boundedness of the multiplicities

This section is devoted to the proof of Theorem B. We will prove (HB) $\Rightarrow$ (1.1) based on the general formula (2.8) on upper bounds of multiplicities (see Theorem 2.4). The opposite implication (1.2) $\Rightarrow$ (BB) (or (1.3) $\Rightarrow$ (BB) when H is reductive) by using Theorem 3.1 on lower bounds.

We begin with the following uniform estimate of multiplicities of irreducible representations occurring in principal series representations as subquotients for which there is, to our knowledge, no direct proof in the literature. So we will give its proof in the appendix (see Section 6.2).

Proposition 4.1. *There exits a constant N depending only on G such that*

$$[I_P^G(\zeta) : \pi] \leq N \quad \textit{for any } \pi \in \widehat{G}_{\mathrm{ad}} \textit{ and for any } \zeta \in \widehat{L}_{\mathrm{f}}.$$

Retain the notation of Section 2. In particular, B is the Borel subgroup of G_c with the Lie algebra $\mathfrak{b}$ given by the positive system $\Sigma(\mathfrak{j})^+$. Then $\mathfrak{b}$ is contained in the complexified Lie algebra $\mathfrak{p}_c$ of the minimal parabolic subgroup $P = LN$ of G.

Lemma 4.2. *If H_c acts on G_c/B with an open orbit, then there exists $g \in G$ such that $H_c gB$ is open in G_c and that HgP is open in G. In particular,* (HB) $\Rightarrow$ (HP).

Proof. Put $G_c' = \{g \in G_c : \mathrm{Ad}(g)\mathfrak{h}_c + \mathfrak{b} \neq \mathfrak{g}_c\}$. Then G_c' is a proper closed analytic subset of the complex manifold G_c. Hence $G \not\subset G_c'$ and there exists $g \in G$ with $\mathrm{Ad}(g)\mathfrak{h}_c + \mathfrak{b} = \mathfrak{g}_c$, which implies $\mathrm{Ad}(g)\mathfrak{h} + \mathrm{Lie}(P) = \mathfrak{g}$. ☐

Suppose that H_c has an open orbit on G_c/B. Replacing H by $g^{-1}Hg$ in Lemma 4.2, we may assume that H_cB is open in G_c and HP is open in G. Then we apply Theorem 2.4 and Proposition 4.1 with $\bar{P}$ replaced by P. Thus we have shown

$$c_{\mathfrak{g},K}(\pi, \mathrm{Ind}_H^G \tau) \leq N \#(W(\mathfrak{t}) \backslash W(\mathfrak{j}; d\pi)) \sum_{\substack{\zeta \in \widehat{L}_{\mathrm{f}} \\ d\zeta = d\pi}} c_{H \cap P(V_\zeta, V_{\tau,P})}$$

for any $\pi \in \widehat{G}_{\mathrm{ad}}$ with infinitesimal character $d\pi$ and for any $(\tau, V_\tau) \in \widehat{H}_{\mathrm{f}}$. Now the implication (HB) $\Rightarrow$ (1.1) in Theorem B follows from Proposition 4.3 below on finite dimensional representations.

Let P_0 be the identity component of P, J the Cartan subgroup of G with Lie algebra $\mathfrak{j}$, and $Z(G)$ the center of G. Let D be the maximal dimension of the irreducible representations of J. Note that $D \leq \#(J/Z(G)\exp\mathfrak{j}) = \#(P/Z(G)P_0)$ and $D = 1$ if G is linear.

Proposition 4.3. *Assume that H_cB is open in G_c and that HP is open in G. For any $(\tau, V_\tau) \in \widehat{H}_{\mathrm{f}}$ and $(\zeta, V_\zeta) \in \widehat{L}_{\mathrm{f}}$ we have $c_{H\cap P}(V_\zeta, V_{\tau,P}) \leq D \cdot \dim\tau$.*

Proof. It follows from $\mathfrak{g}_c = \mathfrak{h}_c + \mathfrak{b}$ and $\mathfrak{b} \subset \mathfrak{p}_c$ that

$$\mathfrak{p}_c = (\mathfrak{h}_c \cap \mathfrak{p}_c) + \mathfrak{b}. \tag{4.1}$$

Let $\widetilde{P}_c$ be the connected and simply connected complex Lie group with Lie algebra $\mathfrak{p}_c$. We write $(H \cap P)_c$ and $\widetilde{B}$ for the connected subgroups of $\widetilde{P}_c$ with Lie algebra $\mathfrak{h}_c \cap \mathfrak{p}_c$ and $\mathfrak{b}$, respectively. Then the P-module ζ^* uniquely corresponds to irreducible representations ζ_1 of J and ζ_o of P_0 by the natural map $J \times P_o \ni (j,p) \mapsto jp \in P$ and hence ζ^* is isomorphic to the direct sum of $\dim\zeta_1$ copies of $\mathcal{O}(\widetilde{P}_c/\widetilde{B}, \mathcal{L}_\lambda)$ as $\mathfrak{p}_c$-modules. Here $\mathcal{L}_\lambda$ is the $\widetilde{P}_c$-homogeneous holomorphic line bundle over $\widetilde{P}_c/\widetilde{B}$ associated with a suitable character λ of B such that the space of global holomorphic sections, denoted by $\mathcal{O}(\widetilde{P}_c/\widetilde{B}, \mathcal{L}_\lambda)$, corresponds to the Borel–Weil realization of ζ_o. Note that $\dim\zeta_1 \leq D$. Passing to the space of fixed vectors under the diagonal action of $H \cap P$ on $V_{\tau,P} \otimes V_\zeta$, we have

$$(V_{\tau,P} \otimes V_\zeta)^{H\cap P} \subset \bigoplus^{\dim\zeta_1} \Big(V_{\tau,P} \otimes \mathcal{O}(\widetilde{P}_c/\widetilde{B}, \mathcal{L}_\lambda)\Big)^{\mathfrak{h}_c\cap\mathfrak{p}_c}.$$

Since $(H\cap P)_c$ acts on $\widetilde{P}_c/\widetilde{B}$ with an open orbit by (4.1), $\dim(V_{\tau,P} \otimes V_\zeta)^{H\cap P} \leq \dim\zeta_1 \cdot \dim\tau$ because a holomorphic function on a connected complex manifold is uniquely determined by its restriction to an open subset. Hence Proposition 4.3 is proved. □

Thus we have completed the proof of the implication (HB) ⇒ (1.1) in Theorem B.

Remark 4.4. Let G be real algebraic (not necessarily reductive), σ an involution and $H = G^\sigma$. R. Lipsman proved that the multiplicity of the abstract Plancherel formula for G/H is uniformly bounded under the hypothesis that this statement is true in the reductive case ([19, Theorem 7.3]). Theorem B shows that his hypothesis is true because there always exists an open H_c-orbit on G_c/B for any complex reductive symmetric pair (G_c, H_c).

Let us prove the remaining implication in Theorem B, namely, (1.2) $\Rightarrow$ (HB) (or (1.3) $\Rightarrow$ (HB) when H is reductive).

Let N be the constant in Proposition 4.1. Then, for any $\pi \in \widehat{G}_{\mathrm{ad}}$, $\zeta \in \widehat{L}_{\mathrm{f}}$, and $\tau \in \widehat{H}_{\mathrm{f}}$, we have

$$c_{\mathfrak{g},K}(I_P^G(\zeta), \operatorname{Ind}_H^G \tau) \le N c_{\mathfrak{g},K}(\pi, \operatorname{Ind}_H^G \tau).$$

Therefore the conditions (1.2) and (1.3) imply

$$\sup_{\substack{\tau \in \widehat{H}_{\mathrm{f}} \\ \dim \tau = 1}} \sup_{\zeta \in \widehat{L}_{\mathrm{f}}} c_{\mathfrak{g},K}(I_P^G(\zeta), \operatorname{Ind}_H^G \tau) < \infty,$$

$$\sup_{\zeta \in \widehat{L}_{\mathrm{f}}} c_{\mathfrak{g},K}(I_P^G(\zeta), C^\infty(G/H)) < \infty,$$

respectively. Applying Theorem 3.1 with $m = 1$, we get

$$\sup_{\substack{\tau \in \widehat{H}_{\mathrm{f}} \\ \dim \tau = 1}} \sup_{\sigma \in \widehat{G}_{\mathrm{f}}} c_H(\sigma, \tau) < \infty,$$

$$\sup_{\sigma \in \widehat{G}_{\mathrm{f}}} c_H(\sigma, \mathbf{1}) < \infty,$$

respectively. Hence, the implication (1.2) $\Rightarrow$ (HB) (or (1.3) $\Rightarrow$ (HB) when H is reductive) reduces to the implication (iii)$'$ $\Rightarrow$ (i) (or (iv)$'$ $\Rightarrow$ (i)) in the following classical results on finite dimensional representations:

Lemma 4.5 ([28])**.** *Let H_c be an algebraic subgroup of a complex semisimple Lie group G_c. In what follows $\widehat{G}_{\mathrm{alg}}$, $\widehat{H}_{\mathrm{alg}}$ denote the set of irreducible algebraic finite dimensional irreducible representations of G_c, H_c, respectively. Then the following five conditions on the pair (G_c, H_c) are equivalent:*

(i) *There exists an open H_c-orbit on G_c/B.*

(ii) $c_H(\sigma, \tau) \le \dim \tau$ *for any* $\sigma \in \widehat{G}_{\mathrm{alg}}$ *and* $\tau \in \widehat{H}_{\mathrm{alg}}$.

(ii)$'$ $\displaystyle\sup_{\tau \in \widehat{H}_{\mathrm{alg}}} \sup_{\sigma \in \widehat{G}_{\mathrm{alg}}} \frac{1}{\dim \tau} c_H(\sigma, \tau) < \infty.$

(iii) $c_H(\sigma, \tau) \le 1$ *for any* $\sigma \in \widehat{G}_{\mathrm{alg}}$ *and* $\tau \in \widehat{H}_{\mathrm{alg}}$ *such that* $\dim \tau = 1$.

(iii)$'$ $\displaystyle\sup_{\substack{\tau \in \widehat{H}_{\mathrm{alg}} \\ \dim \tau = 1}} \sup_{\sigma \in \widehat{G}_{\mathrm{alg}}} c_H(\sigma, \tau) < \infty.$

Furthermore, if H is reductive, then they are also equivalent to:

(iv) $c_H(\sigma, \mathbf{1}) \le 1$ *for any* $\sigma \in \widehat{G}_{\mathrm{alg}}$.

(iv)′ $\sup_{\sigma \in \widehat{G}_{\mathrm{alg}}} c_H(\sigma, \mathbf{1}) < \infty$.

Proof. The following implications are obvious:

$$\begin{array}{ccccc} \text{(ii)} & \Rightarrow & \text{(iii)} & \Rightarrow & \text{(iv)} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{(ii)}' & \Rightarrow & \text{(iii)}' & \Rightarrow & \text{(iv)}' \end{array}$$

The implication (i) $\Rightarrow$ (ii) follows easily from the Borel–Weil theorem. The non-trivial part is (iii) $\Rightarrow$ (i) (or (iv) $\Rightarrow$ (i)), which was proved in Vinberg–Kimelfeld [28].

Let us show the remaining (and easy) implication (iii)′ $\Rightarrow$ (iii) (or (iv)′ $\Rightarrow$ (iv)). Suppose $c_H(\sigma, \tau) \ge 2$ for some $\sigma \in \widehat{G}_{\mathrm{alg}}$ and $\tau \in \widehat{H}_{\mathrm{alg}}$ with $\dim \tau = 1$. Then we can find two linearly independent highest weight vectors f_1, $f_2 \in \mathcal{O}(G_c)$ such that $f_j(b^{-1}gh) = \chi_\sigma(b)\tau(h^{-1})f_j(g)$ $(j = 1, 2)$ for any $b \in B$, $h \in H_c$, and $g \in G_c$ where χ_σ corresponds to a highest weight of σ. We claim that $f_1^i f_2^{N-i}$ $(0 \le i \le N)$ are linearly independent. Indeed, suppose $a_0 f_1^N + a_1 f_1^{N-1} f_2 + \cdots + a_N f_2^N = 0$ is a linear dependence. Let λ be a zero of the equation $a_0 t^N + a_1 t^{N-1} + \cdots + a_N = 0$. Since the ring $\mathcal{O}(G_c)$ has no divisor, we have $f_1 - \lambda f_2 = 0$, which contradicts to the linear independence of f_1 and f_2. Therefore, we have $c_H(\sigma_N, \tau^N) \ge N + 1$ where $\sigma_N \in \widehat{G}_{\mathrm{alg}}$ is defined to have a highest weight χ_σ^N. Hence (iii)′ $\Rightarrow$ (iii) is shown. The implication (iv)′ $\Rightarrow$ (iv) is immediate by putting $\tau = \mathbf{1}$. $\square$

We have thus completed the proof of Theorem B.

5 Restriction of irreducible representations

In this section we discuss the restriction of an admissible irreducible representation π of a semisimple Lie group with respect to a reductive subgroup H, and give a proof of Theorems C and D on geometric criteria for finiteness and boundedness of the dimension of $\mathrm{Hom}_H(\pi|_H, \tau)$, the space of continuous H-homomorphisms for $\tau \in \widehat{H}_{\mathrm{ad}}$.

In dealing with the restrictions of admissible representations which are not necessarily unitary, we work mostly in the framework of smooth representations. We begin with an elementary observation:

Lemma 5.1. *Suppose* $(\pi, V_\pi) \in \widehat{G}_{\mathrm{ad}}$ *and* $(\tau, V_\tau) \in \widehat{H}_{\mathrm{ad}}$. *Then we have a natural injective map*

$$\mathrm{Hom}_H(V_\pi, V_\tau) \to \mathrm{Hom}_H(V_\pi^\infty, V_\tau^\infty), \quad \varphi \mapsto \varphi|_{V_\pi^\infty}.$$

Proof. Let $\varphi : V_\pi \to V_\tau$ be a continuous H-homomorphism. If v is a smooth vector of V_π as a representation of G, then v is a smooth vector for the representation $\pi|_H$ of the subgroup H, and consequently, so is $\varphi(v)$ for τ. Since V_π^∞ is dense in V_π, $\varphi \mapsto \varphi|_{V_\pi^\infty}$ is injective. □

Let ΔH denote the diagonal subgroup $\{(h,h) : h \in H\}$ in $G \times H$. The next lemma reduces the problem of the restriction to a problem on the induced representation for which we have already solved in Sections 2 and 3:

Lemma 5.2. *For any* $\pi \in \widehat{G}_\infty$ *and* $\tau \in \widehat{H}_\infty$, *there is a natural bijection*

$$\mathrm{Hom}_H(\pi|_H, \tau) \simeq \mathrm{Hom}_{G\times H}(\pi \times \sigma, C^\infty(G \times H/\Delta H)).$$

Here, τ^* *is the contragredient representation of* H *on the continuous dual (the space of distribution vectors), and* σ *denotes its smooth representation* $(\tau^*)^\infty$.

Proof. We write V_π, V_τ, and V_σ for the representation spaces of the smooth representations π, τ, and σ, respectively.

Suppose $\varphi : V_\pi \to V_\tau$ is a continuous H-homomorphism. We define a continuous map $\Phi : V_\pi \times V_\sigma \times G \times H \to \mathbb{C}$ by

$$\Phi(v, u; g, h) := \langle \varphi(\pi(g^{-1})v), \sigma(h^{-1})u \rangle.$$

Then the induced map $(v, u) \mapsto \Phi(v, u; \cdot, \cdot)$ gives a continuous $(G \times H)$-homomorphism from $V_\pi \times V_\sigma$ to $C^\infty(G \times H/\Delta H)$.

Conversely, suppose $\Psi : V_\pi \times W_\sigma \to C^\infty(G \times H/\Delta H)$ is a continuous $(G \times H)$-homomorphism. Then the linear map

$$\psi : V_\pi \to W_\sigma^*, \ v \mapsto \Psi(v, \cdot)(e, e)$$

is a continuous H-homomorphism, and therefore, its image is contained in the subspace $(W_\sigma^*)^\infty$ of W_σ^*. Since every smooth representation τ of H is reflexive, i.e. $(W_\sigma^*)^\infty \simeq V_\tau$, we have now shown Lemma. □

Combining Lemma 5.1 with Lemma 5.2, we get

$$
\begin{aligned}
\dim \operatorname{Hom}_H(\pi|_H, \tau) &\leq \dim \operatorname{Hom}_H(\pi^\infty|_H, \tau^\infty) \\
&= \dim \operatorname{Hom}_{G\times H}(\pi^\infty \times (\tau^*)^\infty, C^\infty(G \times H/\Delta H)).
\end{aligned} \tag{5.1}
$$

Now Theorems C and D follow from (5.1) and Theorems A and B in light of the following elementary observation:

Lemma 5.3. 1) *The condition* (PP) *holds for the pair* (G, H) *if and only if the condition* (HP) *holds for* $(G \times H, \Delta H)$.
2) *The condition* (BB) *holds for* (G, H) *if and only if the condition* (HB) *holds for* $(G \times H, \Delta H)$.

Proof. 1) $P \times P_H$ is a minimal parabolic subgroup of $G \times H$. The claim follows from the natural bijection $(P \times P_H)\backslash(G \times H)/\Delta H \simeq P\backslash G/P_H$. 2) Similarly, $B \times B_H$ is a Borel subgroup of $G_c \times H_c$, and the claim follows from the bijection $(B \times B_H)\backslash(G_c \times H_c)/\Delta H_c \simeq B\backslash G_c/B_H$. □

In the case where π is unitary, we can decompose the restriction $\pi|_H$ into the direct integral of irreducible unitary representations of H, and such a decomposition (*branching law*) is unique as H is of type I in the sense of von Neumann algebras. We denote by $\widehat{G}$ the set of (unitary) equivalence classes of irreducible unitary representations of G. For $(\pi, V_\pi) \in \widehat{G}$, $(\tau, W_\tau) \in \widehat{H}$, $\varphi \in \operatorname{Hom}_H(\tau, \pi|_H)$ gives an irreducible summand $\varphi(W_\tau)$ in V_π. As an immediate corollary of Theorems C and D, we give an upper bound of the multiplicity in the discrete part:

Theorem 5.4. *Suppose* (G, H) *is a pair of reductive Lie groups.*
1) *If there is an open* P_H*-orbit on* G/P, *then* $\dim \operatorname{Hom}_H(\tau, \pi|_H) < \infty$ *for any* $\pi \in \widehat{G}$ *and* $\tau \in \widehat{H}$.
2) *If there is an open* B_H*-orbit on* G_c/B, $\sup_{\pi\in\widehat{G},\tau\in\widehat{H}} \dim \operatorname{Hom}_H(\tau, \pi|_H) < \infty$.

Proof of Theorem 5.4. Since the adjoint map gives an anti-linear bijection

$$\operatorname{Hom}_H(V_\tau, V_\pi) \simeq \operatorname{Hom}_H(V_\pi, V_\tau),$$

Theorem 5.4 follows from Theorems B and D. □

Remark 5.5. 1) Theorem 5.4 2) was announced in this form in [12, Part II, Remark 2.10]. See [1, 14, 26] for recent results without unitarity.
2) If $H = K$ a maximal compact subgroup of G, then the assumption of Theorem 5.4 1) is obviously satisfied because $P_H = K$ and $KP = G$. In this case $\dim \tau < \infty$ for any $\tau \in \widehat{K}$. This simplest case gives an analytic proof to the celebrated result of Harish-Chandra asserting that *any irreducible unitary representation is admissible* (using a theorem of I. Segal on the existence of infinitesimal characters of irreducible unitary representations).
3) Even if (PP) fails, it may happen that $\dim \mathrm{Hom}_H(\tau, \pi|_H) < \infty$ for any $\tau \in \widehat{H}$ for a specific triple (π, G, H). This was studied in details in [12] when the decomposition is discretely decomposable.

6 Appendix

6.1 Proof of Lemma 3.5

Lemma 6.1. *Let H_c be a complex algebraic group acting on a smooth complex variety X by $\Psi: H_c \times X \ni (g,x) \mapsto gx \in X$. Then there exists a locally closed submanifold Y of X in the Zariski topology such that the following two conditions holds:*
1) *$\Psi|_{H_c \times Y}$ is a submersion.*
2) *$\#\big(Y \cap \Psi^{-1}_{H_c \times Y}(y)\big)$ is finite and does not depend on $y \in Y$.*

Proof. Let ℓ be the minimal codimension of the submanifold $H_c x$ for $x \in X$. Fix $p \in X$ such that the codimension of $H_c p$ equals ℓ. Let Y be an ℓ-dimensional locally closed submanifold of X through p such that $d\Psi|_{H_c \times Y}$ is surjective at (e,p). By shrinking Y if necessary, we may assume that $d\Psi|_{H_c \times Y}$ surjects $T_y Y$ at (e,y) for all $y \in Y$. Since the surjectivity of $d\Psi|_{H_c \times Y}$ at (e,y) implies that of $d\Psi|_{H_c \times Y}$ at (h,y) for any $h \in H$, $\Psi|_{H_c \times Y}$ is a submersion. Consider a locally closed subvariety

$$\tilde{Y} = (\mathrm{pr} \times \mathrm{id}_Y) \circ (\Psi|_{H_c \times Y} \times \mathrm{id}_Y)^{-1}(\Delta Y)$$

of $Y \times Y$, where pr is the natural projection map of $G \times Y$ onto Y and $\Delta Y = \{(y,y) \in Y \times Y : y \in Y\}$. By definition, $(x,y) \in \tilde{Y}$ if and only if $H_c x = H_c y$. Since the fiber of the map $\pi: \tilde{Y} \ni (x,y) \mapsto x \in Y$ is discrete, there exist a positive number m and a Zariski open subset Y' of Y such that $\pi|_{\pi^{-1}(Y')}$ is an m-fold covering map of Y'. Then we have the lemma by replacing Y by Y'. □

Lemma 6.2. *Suppose we are in the setting of Lemma 6.1. If H and M are real forms of H_c and X such that $H \cdot M \subset M$, then there exists a locally closed submanifold N of M in the usual topology satisfying the following two conditions:*
1) $\Psi|_{H\times N}$ is a submersion of $H \times N$ to M,
2) $Hx \neq Hy$ for any $x, y \in N$ with $x \neq y$.

Proof. Put $N' = M \cap Y$. Owing to Lemma 6.1 2), the cardinality

$$n = \sup_{x\in N'} \#\{y \in N' : Hx = Hy\}$$

is finite and so we can find n different points $p_1, \ldots, p_n$ of N' such that $Hp_1 = \cdots = Hp_n$. Let U_i be open neighborhoods of p_i in N' which do not meet each other. Then the open subset $N = \{p \in U_1 : Hp \cap U_i \neq \emptyset \text{ for } i = 1, \ldots, n\}$ of N' is the required one. □

The following lemma in the non-algebraic setting may also be useful for Theorem 3.1.

Lemma 6.3. *Let H be a Lie group acts on a manifold M. Suppose there exists a locally closed submanifold N of M such that the map $\Psi : H \times M \ni (h, x) \mapsto hx \in M$ satisfies* Lemma 6.2 1) *and*
2)′ $m_N(x) < \infty$ for any $x \in N$.
Here we set $m_N(x) := \#\{y \in N : Hy = Hx\}$. Then the conditions Lemma 6.2 1) *and* 2) *are satisfied by shrinking N if necessary.*

Proof. Put $U_i = \{x \in N : m_N(x) > i\}$ for $i = 1, 2, \ldots$. Then U_i are open subsets of N because $\Psi(H \times U) \cap N$ is open in N for any open subset U of N. Put $V_i = N \backslash U_i$. Since $\bigcup_i V_i = N$ by our assumption, Baire's category theorem says that there exists V_m having an inner point under the induced topology of N. Replacing N by the interior of V_m and using the same argument in the proof of Lemma 6.2, we have Lemma 6.3. □

6.2 Proof of Proposition 4.1

We shall prove a uniform estimate of the multiplicity of irreducible representations occurring in a principal series representation.

Suppose we are in the setting of Section 4. Let $\alpha_1, \ldots, \alpha_n$ be the fundamental system in $\Sigma(\mathfrak{j})^+$ and $\omega_1, \cdots, \omega_n$ the corresponding fundamental weights. By taking a covering group of G if necessary, we may assume that G is the real form of the simply connected complex Lie group G_c or its covering group, so that the fundamental representation V_i with the highest weight ω_i lifts to G. For $\lambda \in \mathfrak{j}_c^*$ we put

$$\lambda = \sum_{i=1}^{n} \Lambda_i(\lambda)\omega_i$$

and define

$$\operatorname{Re}\lambda = \sum_{i=1}^{n} (\operatorname{Re}\Lambda_i(\lambda))\omega_i.$$

We will review the Jantzen–Zuckerman translation principle. Let $\mathcal{F}_\lambda(\mathfrak{g}, K)$ be the category of $(\mathfrak{g}, K)$-modules of finite length with a generalized infinitesimal character χ_λ. After conjugation by the Weyl group if necessary, we may assume λ satisfies

$$\operatorname{Re}\langle\lambda, \alpha\rangle \leq 0 \quad \text{for } \alpha \in \Sigma(\mathfrak{j})^+. \tag{6.1}$$

For $V \in \mathcal{F}_\lambda(\mathfrak{g}, K)$ we define $\Phi_\lambda^i(V) := p_{\lambda+\omega_i}(V \otimes V_i)$. Here $p_{\lambda+\omega_i}$ is the projection map to the primary component with generalized infinitesimal character $\lambda + \omega_i$. Then Φ_λ^i is an exact functor from $\mathcal{F}_\lambda(\mathfrak{g}, K)$ to $\mathcal{F}_{\lambda+\omega_i}(\mathfrak{g}, K)$. Similarly, we define a functor from $\mathcal{F}_{\lambda+\omega_i}(\mathfrak{g}, K)$ to $\mathcal{F}_\lambda(\mathfrak{g}, K)$ by $\Psi_\lambda^i(W) := p_\lambda(W \otimes V_i^*)$, where V_i^* is the contragredient representation of V_i.

Let $(\zeta, V_\zeta) \in \widehat{L}_{\mathrm{f}}$. We write $d\zeta \in \mathfrak{j}_c^*$ for the highest weight with respect to $\Sigma(\mathfrak{t})^+$, and take $w_o \in W(\mathfrak{j})$ such that $\lambda := w_o d\zeta$ satisfies (6.1). Assume $\operatorname{Re}\Lambda_i(\lambda) < -1$ for some i. This assumption assures that λ and $\lambda + \omega_i$ are equisingular, namely, $\langle\lambda, \alpha\rangle = 0 \Leftrightarrow \langle\lambda + \omega_i, \alpha\rangle = 0$ for $\alpha \in \Sigma(\mathfrak{j})$. Then we have an isomorphism of $(\mathfrak{g}, K)$-modules:

$$\Psi_\lambda^i(I_P^G(\zeta')) \simeq I_P^G(\zeta) \ \text{ and } \ I_P^G(\zeta') \simeq \Phi_\lambda^i(I_P^G(\zeta)). \tag{6.2}$$

Here $(\zeta', V_{\zeta'}) \in \widehat{L}_{\mathrm{f}}$ is the unique representation such that $V_{\zeta',P}$ occurs as a subquotient of $V_{\zeta,P} \otimes V_i$ and satisfies $w_o d\zeta' = \lambda + \omega_i$. Thanks to [11, Theorem 7.232], Φ_λ^i induces an equivalence of categories between $\mathcal{F}_\lambda(\mathfrak{g}, K)$ and $\mathcal{F}_{\lambda+\omega_i}(\mathfrak{g}, K)$. In particular, Φ_λ^i sends (non-zero) irreducible $(\mathfrak{g}, K)$-modules to (non-zero) irreducible $(\mathfrak{g}, K)$-modules and we have

$$\begin{aligned}\operatorname{Hom}_{(\mathfrak{g},K)}(\pi, I_P^G(\zeta)) &\simeq \operatorname{Hom}_{(\mathfrak{g},K)}(\Phi_\lambda^i(\pi), \Phi_\lambda^i(I_P^G(\zeta))) \\ &\simeq \operatorname{Hom}_{(\mathfrak{g},K)}(\Phi_\lambda^i(\pi), I_P^G(\zeta'))\end{aligned}$$

for any $(\pi, V) \in \mathcal{F}_\lambda(\mathfrak{g}, K)$. Here we use (6.2) for the second equality. Hence applying Φ_λ^i successively, we may assume

$$|\operatorname{Re} d\zeta| \leq C$$

in order to prove Proposition 4.1, where $|\cdot|$ is the norm induced from the Killing form and $C := |\sum_{i=1}^n \omega_i|$.

Now we recall Vogan's results on minimal K-type theory. We take a Cartan subalgebra $\tilde{\mathfrak{t}}$ of $\mathfrak{k}$ and fix a positive system $\Delta^+(\mathfrak{k}_c, \tilde{\mathfrak{t}}_c)$. We write $\delta_K \in \sqrt{-1}\tilde{\mathfrak{t}}^*$ for half the sum of elements in $\Delta^+(\mathfrak{k}_c, \tilde{\mathfrak{t}}_c)$. If $\mu \in \tilde{\mathfrak{t}}_c^*$ is the highest weight of a K-type τ we define $\|\tau\| := |\mu + 2\delta_K|$, where $|\cdot|$ denotes the norm in $\sqrt{-1}\tilde{\mathfrak{t}}^*$ induced from the Killing form. A minimal K-type of the $(\mathfrak{g}, K)$-module (π, V) is a K-type τ for which $|\tau|$ is minimal among all K-types occurring in π. It follows from [11, Theorem 10.26] that there exists a constant C' depending only on $\mathfrak{g}$ with the following property: if π is a $(\mathfrak{g}, K)$-module with infinitesimal character λ, then

$$|\operatorname{Re}(\lambda)| \geq \|\tau\| - C'.$$

Let N be the maximal dimension of $\tau \in \widehat{K}$ among all K-types τ with $\|\tau\| \leq C + C'$. We remark that N depends only on the Lie algebra $\mathfrak{g}$. For $\pi \in \widehat{G}_{\mathrm{ad}}$, let τ be one of its minimal K-types. Because τ occurs in π with multiplicity one, we have

$$[\pi : I_P^G(\zeta)] \leq \dim \operatorname{Hom}_K(\tau, I_P^G(\zeta)).$$

Then the right-hand side equals $\dim \operatorname{Hom}_M(\tau_{|M}, \zeta_{|M})$ by the Frobenius reciprocity theorem. Since $\dim \operatorname{Hom}_M(\tau_{|M}, \zeta_{|M}) \leq N$, we have proved Proposition 4.1. □

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